CHAPTER 4
PRIME IDEALS OF \( (\sigma, \delta) \)-RINGS

In this chapter we discuss some special types of prime ideals that play a key role in the notions introduced. These type of prime ideals include completely prime ideals and minimal prime ideals. Minimal prime ideals play an important role in understanding rings and modules. We know that each minimal prime ideal of a ring without nonzero nilpotent elements is completely prime, [32]. In the present chapter we find relation between minimal prime ideals and completely prime ideals of a \( (\sigma, \delta) \)-ring and weak \( (\sigma, \delta) \)-rigid ring (Theorem (4.2.1) and (4.3.1)). In Theorem (4.2.4), we prove that a minimal prime ideal of a \( (\sigma, \delta) \)-ring which is \( \sigma \)-stable (Definition (4.1.4)) is \( \delta \)-invariant (Definition (4.1.4)). The set of minimal prime ideals of \( R \) is denoted by Min.Spec\( (R) \).

4.1 Minimal Prime Ideals and Completely Prima Ideals

Recall the following:

**Definition 4.1.1:** An ideal \( P \) of a ring \( R \) is said to be a completely prime ideal if \( ab \in P \) implies that \( a \in P \) or \( b \in P \) for \( a, b \in R \). (Goodearl and Warfield, 2004, [23]).

In commutative sense completely prime ideal and prime ideal have the same meaning. We also note that a completely prime ideal of a ring \( R \) is a prime ideal, but the converse need not be true. The following example shows that a prime ideal need not be a completely prime ideal.

**Example 4.1.2:** Let \( R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix} = M_2(\mathbb{Z}) \) be a ring. If \( p \) is a prime number, then the ideal \( P = M_2(p\mathbb{Z}) \) is a prime ideal of \( R \). But is not
a completely prime ideal, since for \(a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\) and \(b = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\) we have \(ab \in P\), even though \(a \notin P\) and \(b \notin P\). (Example (1.1) of [7]).

There are examples of rings (non-commutative) in which prime ideals are completely prime ideals.

**Example 4.1.3:** Let \(R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix}\). Then

\[
P_1 = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & 0 \end{pmatrix}, P_2 = \begin{pmatrix} 0 & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix} \text{ and } P_3 = \begin{pmatrix} 0 & \mathbb{Z} \\ 0 & 0 \end{pmatrix}
\]

are prime ideals of \(R\). Now all these are completely prime ideals also. (Example (1.2) of [7]).

In [45], Mulvey defines a ring \(R\) to be a Gelfand ring provided that for any distinct maximal right ideals \(M\) and \(M'\) of \(R\) there exists elements \(a \notin M\), \(a' \notin M'\) of the ring for which \(aRa' = 0\). Note that every Gelfand ring is a completely prime ideal as in [53].

**Definition 4.1.4:** Let \(R\) be a ring, \(\sigma\) an endomorphism of \(R\) and \(\delta\) a \(\sigma\)-derivation of \(R\). An ideal \(I\) of a ring \(R\) is called \(\sigma\)-stable if \(\sigma(I) = I\) and it is called \(\delta\)-invariant if \(\delta(I) \subseteq I\). (Bhat, 2010, [8])

**Example 4.1.5:** Let \(R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix}\). Then \(I = \begin{pmatrix} 0 & \mathbb{Z} \\ 0 & 0 \end{pmatrix}\) is an ideal of \(R\).

Define an automorphism \(\sigma : R \to R\) by

\[
\sigma \left( \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \right) = \begin{pmatrix} a & -b \\ 0 & c \end{pmatrix} \text{ for } a, b, c \in \mathbb{Z}.
\]

Also let \(\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \in I\) for \(b \in \mathbb{Z}\). Now
\[
\sigma\left(\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & -b \\ 0 & 0 \end{pmatrix}.
\]

Therefore, \(\sigma(I) = I\). Hence \(I\) is \(\sigma\)-stable.

Define \(\delta : R \to R\) by

\[
\delta(A) = A - \sigma(A) \text{ for } A \in R.
\]

Clearly, \(\delta\) is a \(\sigma\)-derivation of \(R\).

Now let \(A = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \in I\).

Further \(\delta\left(\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} - \sigma\left(\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}\right)\)

\[
= \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2b \\ 0 & 0 \end{pmatrix} \subseteq I.
\]

Hence \(\delta(I) \subseteq I\). Thus \(I\) is \(\delta\)-invariant.

**Example 4.1.6:** Let \(R = \mathbb{Z} \times \mathbb{Z}\) be a ring. Define \(\sigma : R \to R\) by

\[
\sigma((a, b)) = (b, a) \text{ for } a, b \in \mathbb{Z}.
\]

Then \(\sigma\) is an automorphism of \(R\). But it is not \(\sigma\)-stable, because for \(I = \mathbb{Z} \times \{0\}\), \(\sigma(I) = \{0\} \times \mathbb{Z} \neq I\).

**Definition 4.1.7:** A minimal prime ideal in a ring \(R\) is any prime ideal of \(R\) that does not properly contain any other prime ideal. (Goodearl and Warfield, 2004, [23])

**Example 4.1.8:** In example (4.1.3) (discussed above), \(P_3 = \begin{pmatrix} 0 & \mathbb{Z} \\ 0 & 0 \end{pmatrix}\) is the minimal prime ideal.

If \(R\) is a prime ring, then \(0\) is a minimal prime ideal of \(R\) and it is the only one. In Proposition (3.3) of [19], it has been shown that any prime ideal \(U\)
in a ring $R$ contains a minimal prime ideal. Further it has been proved that there exists only finitely many minimal prime ideals in a Noetherian ring $R$ and there is a finite product of minimal prime ideals (repetition allowed) that equals zero. An example of a ring which has infinitely many minimal prime ideals is:

**Example 4.1.9:** Let $X$ be an infinite set, $K$ a field and $R$ the ring of all functions from $X$ to $K$. For $x \in X$, let $P_x$ be the set of those functions in $R$ which vanish at $x$. Then each $P_x$ is a minimal prime ideal of $R$. (Goodearl and Warfield, 2004, [23]).

Furthermore there are examples of rings in which minimal prime ideals are completely prime ideals. For example a reduced ring (Theorem (2.4) of [32]) i.e., a ring without nilpotent elements (We give the proof in Theorem (4.1.12) for ready reference). To prove this we need the following notations and Propositions:

Let $R$ be a ring and $P$ a prime ideal of $R$, then $0_P = \{ r \in R \mid ra = 0, \exists a \notin P \}$. Also for a non-empty subset $S$ of a ring $R$, let $S^r = \{ r \in R \mid sr = 0 \text{ for every } s \in S \}$, $S^l = \{ r \in R \mid rs = 0 \text{ for every } s \in S \}$ and if $S^r = S^l$ then let $S^\perp = S^r$.

**Proposition 4.1.10:** Let $R$ be a ring without nilpotent elements and let $x$ be a nonzero element of $R$. Then $\{x\}^r$ is a two-sided ideal of $R$, $\{x\}^r = \{x\}^l$, $x \notin \{x\}^l$, $R/\{x\}^\perp$ has no nilpotent elements and if $r \in R$ and $rx \in \{x\}^l$ then $r \in \{x\}^l$.

**Proof.** Proposition (2.1) of [32]. 

**Proposition 4.1.11:** If $P$ is a prime ideal of a ring $R$ without nilpotent elements then $0_P = \{ r \in R \mid ra = 0, \exists a \notin P \}$ is an ideal, $0_P \subseteq P$ and $R/0_P$ is a ring without nilpotent elements.

**Proof.** Proposition (2.3) of [32].
**Theorem 4.1.12:** Let $R$ be a ring without nilpotent elements. Then $P$ is a minimal prime ideal if and only if $P = 0_P$ and in this case, $P$ is a completely prime ideal.

**Proof.** Let $P$ be a minimal prime ideal. Suppose $P \neq 0_P$. Then there is $a \in P$ such that $a \notin 0_P$, since $0_P \subseteq P$ by Proposition (4.1.11). Let $M = R/P$. Then $M$ is an m-system, that is for any $x, y \in M$ there is $r \in R$ such that $xry \in M$. Let $S = \{a, a^2, a^3, \ldots\}$ and let $T = \{r \in R \mid r \neq 0, r = a^{i_0}x_0a^{i_1}x_1\cdots a^{i_n}x_na^{i_{n+1}}\}$ for some non-negative integer $n$ where $x_j \in M$ for $j = 0, 1, 2, ..., n$ and $i_0, i_{n+1}$ are non-negative integers and $i_1, i_2, ..., i_n$ are positive integers. We let $ra^0 = ra^0r$ for any $r \in R$. We will prove that $\Gamma = M \cup S \cup T$ is an m-system. It is clear that $0 \notin \Gamma$. Let $x, y$ be two elements in $\Gamma$. Let $x \in M$. If $y \in M$ or $y \in S$ then clearly there is $r \in R$ such that $xry \in \Gamma$ since $M$ is an m-system and $\{a^n\}^\perp \subseteq P$ for any $n$. For if $xa^n = 0$ for some $n$ then $(xa)(xa)...(xa)$ is zero by Proposition (4.11) and this in turn implies $xa = 0 = nax$ and $a \in 0_P$. This is impossible. Now let $y \in T$. Then $y = a^{i_0}x_0a^{i_1}x_1\cdots a^{i_n}x_na^{i_{n+1}}$ for some $x_i \in M$, $i = 0, 1, ..., n$. Since $M$ is an m-system, there exist $r_0, r_1, r_2, ..., r_n$ such that $xr_0x_0r_1x_1r_2x_2...r_nx_n \in M$. Let $w = xr_0x_0r_1x_1r_2x_2...r_nx_n$. Let $i = i_0 + i_1 + ... + i_{n+1}$. If $wy = 0$, then $wy \in T$ and therefore, $xry \cap \Gamma = \phi$. So suppose $wy = 0$. Then by Proposition (4.1.11), $0 = [(a^i w)(a^i w)...(a^i w)]/(n+2)$ and $a^i w = 0$ and $\{a^i\}^\perp$ is not contained in $P$. This is impossible. A similar argument shows that when $x \in S \cup T$, $xry \cap \Gamma = \phi$. Let $A$ be an ideal of $R$ which is maximal with respect to the property that $\Gamma \cap A = \phi$. Then $A$ is a prime ideal and $A \subseteq P$ and $A \neq P$. This contradicts the minimality of $P$. Thus, $0_P = P$.

Conversely, suppose $0_P = P$ and $P'$ is a prime ideal contained in $P$. Then for any $x \in P$ there exists $a \notin P$ such that $xa = 0 \in P'$. This means that $xRa = 0 \subseteq P'$ by Proposition (4.1.11), and $x \in P'$. Thus $P' = P$ is a minimal prime ideal of $R$. By Proposition (4.1.12), $R/0_P$ is a ring without nilpotent elements. Thus, $R/0_P$ is an integral domain. □
4.2 Minimal Prime Ideals and Completely Prime Ideals of \((\sigma, \delta)\)-Rings

In this section we find a relation between minimal prime ideals and completely prime ideals of an integral domain \(R\), where \(R\) is a \((\sigma, \delta)\)-ring. Recall that a ring \(R\) is called a \((\sigma, \delta)\)-ring if \(a(\sigma(a) + \delta(a)) \in P(R)\) implies that \(a \in P(R)\) for \(a \in R\), where \(P(R)\) is the prime radical of \(R\). We first prove a necessary and sufficient condition for a minimal prime ideal to be a completely prime ideal.

**Theorem 4.2.1:** Let \(R\) be a Noetherian integral domain. Let \(\sigma\) be an automorphism of \(R\) and \(\delta\) a \(\sigma\)-derivation of \(R\). Assume that \(\sigma(U) = U\) and \(\delta(U) \subseteq U\) for every \(U \in \text{Min.Spec}(R)\). Then \(R\) is a \((\sigma, \delta)\)-ring if and only if every \(U \in \text{Min.Spec}(R)\) is completely prime and \(\sigma(U) + \delta(U) = U\).

**Proof.** Let \(R\) be a Noetherian ring such that for each minimal prime \(U\) of \(R\), \(\sigma(U) + \delta(U) = U\) and \(U\) is a completely prime ideal of \(R\). Let \(a \in R\) be such that \(a(\sigma(a) + \delta(a)) \in P(R) = \cap_{i=1}^{n} U_i\), where \(U_i\) are the minimal primes of \(R\). Now for each \(i\), \(a \in U_i\) or \(\sigma(a) + \delta(a) \in U_i\) and \(U_i\) is completely prime. Now \(\sigma(a) + \delta(a) \in U_i = \sigma(U_i) + \delta(U_i)\) which implies that \(a \in U_i\) and hence \(a \in P(R)\). Thus \(R\) is a \((\sigma, \delta)\)-ring.

Conversely, suppose that \(R\) is a \((\sigma, \delta)\)-ring and let \(U = U_1\) be a minimal prime ideal of \(R\). Then by Theorem (2.2.6), \(P(R)\) is completely semiprime. Let \(U_2, U_3, ..., U_n\) be the other minimal primes of \(R\). Suppose that

\[ \sigma(U) + \delta(U) \neq U. \]

Then \(\sigma(U) + \delta(U)\) is also a minimal prime ideal of \(R\). Renumber so that \(\sigma(U) + \delta(U) = U_n\). Let \(a \in \cap_{i=1}^{n-1} U_i\). Then \(\sigma(a) + \delta(a) \in U_n\), and so

\[ a(\sigma(a) + \delta(a)) \in \cap_{i=1}^{n} U_i = P(R). \]
Therefore, \( a \in P(R) \) and thus \( \cap_{i=1}^{n-1} U_i \subseteq U_n \) which implies that \( U_i \subseteq U_n \) for some \( i \neq n \). But this is not possible. Hence \( \sigma(U) + \delta(U) = U \).

Now suppose that \( U = U_1 \) is not completely prime. Then there exists \( a, b \in R/U \) with \( ab \in U \). Let \( c \) be any element of \( b(U_2 \cap U_3 \cap ... \cap U_n)a \). Then \( c^2 \in \cap_{i=1}^n U_i = P(R) \). So \( c \in P(R) \) and thus \( b(U_2 \cap U_3 \cap ... \cap U_n)a \subseteq U \). Therefore, \( bR(U_2 \cap U_3 \cap ... \cap U_n)Ra \subseteq U \) and as \( U \) is prime, \( a \in U, U_i \subseteq U \) for some \( i \neq 1 \) or \( b \in U \). None of these can occur, so \( U \) is completely prime.

For the proof of next Theorem (4.2.4), which is the condition for a minimal prime ideal of a \((\sigma, \delta)\)-ring to be \( \delta \)-invariant, we need the following Propositions:

**Proposition 4.2.2:** Let \( R \) be a Noetherian ring which is also an algebra over \( \mathbb{Q} \). Let \( \delta \) be a derivation of \( R \). Then \( \delta(P(R)) \subseteq P(R) \).

*Proof.* See Proposition (1.1) of [5]. \( \square \)

**Proposition 4.2.3:** Let \( R \) be a 2-primal ring. Let \( \sigma \) be an automorphism of \( R \) and \( \delta \) a \( \sigma \)-derivation of \( R \) such that \( \delta(P(R)) \subseteq P(R) \). If \( P \in \text{Min.Spec}(R) \) is such that \( \sigma(P) = P \), then \( \delta(P) \subseteq P \).

*Proof.* Let \( P \in \text{Min.Spec}(R) \). Now \( P \) is a completely prime ideal, therefore, for any \( a \in P \), there exists \( b \notin P \) such that \( ab \in P(R) \), by Corollary (1.10) of Shin [50]. Now \( \delta(P(R)) \subseteq P(R) \), and therefore \( \delta(ab) \subseteq P(R) \);

i.e.,

\[
\delta(a)\sigma(b) + a\delta(b) \in P(R) \subseteq P.
\]

Now \( a\delta(b) \in P \) implies that \( \delta(a)\sigma(b) \in P \). Now \( \sigma(P) = P \) implies that \( \sigma(b) \notin P \) and since \( P \) is completely prime in \( R \), we have \( \delta(a) \in P \). Hence \( \delta(P) \subseteq P \). \( \square \)

**Theorem 4.2.4:** Let \( R \) be a Noetherian, integral domain which is also an algebra over \( \mathbb{Q} \). Let \( \sigma \) be an automorphism of \( R \) and \( \delta \) a \( \sigma \)-derivation of \( R \) such that \( R \) is a \((\sigma, \delta)\)-ring. If \( P \in \text{Min.Spec}(R) \) is such that \( \sigma(P) = P \), then \( \delta(P) \subseteq P \).
Proof. Let \( P \in \text{Min.Spec}(R) \). Then by Proposition (4.2.2), \( \delta(P(R)) \subseteq P(R) \) and by Theorem (2.2.8), \( R \) is 2-primal. Since \( \sigma(P) = P \), the result follows by Proposition (4.2.3). \( \square \)

4.3 Minimal Prime Ideals And Completely Prime Ideals of Weak \((\sigma, \delta)\)-Rigid Rings

We now characterize minimal prime ideals and completely prime ideals of a ring \( R \), where \( R \) is a weak \((\sigma, \delta)\)-rigid ring. Recall that a ring \( R \) is called a weak \((\sigma, \delta)\)-rigid ring if \( a(\sigma(a) + \delta(a)) \in N(R) \) implies and is implied by \( a \in N(R) \) for \( a \in R \), where \( N(R) \) is the nil radical of \( R \). We begin with the following:

**Theorem 4.3.1:** Let \( R \) be a Noetherian integral domain which is also a \( \mathbb{Q} \)-algebra. Let \( \sigma \) be an automorphism of \( R \) and \( \delta \) a \( \sigma \)-derivation of \( R \). Assume that \( \sigma(U) = U \) and \( \delta(U) \subseteq U \) for every \( U \in \text{Min.Spec}(R) \). Then \( R \) is a 2-primal weak \((\sigma, \delta)\)-rigid ring if and only if every \( U \in \text{Min.Spec}(R) \) is completely prime and \( \sigma(U) + \delta(U) = U \).

**Proof.** It follows by Theorems (4.2.1) and (3.2.9). \( \square \)

We now find a relation between weak \((\sigma, \delta)\)-rigid ring and the set of nilpotent elements of an integral domain in the form of Theorem (4.3.2).

**Theorem 4.3.2:** Let \( R \) be a Noetherian integral domain which is also an algebra over \( \mathbb{Q} \). Let \( \sigma \) be an automorphism of \( R \) and \( \delta \) a \( \sigma \)-derivation of \( R \). Then \( R \) a weak \((\sigma, \delta)\)-rigid ring implies that \( N(R) \) is completely semiprime.

**Proof.** Let \( R \) be a weak \((\sigma, \delta)\)-rigid ring. Then we will show that \( N(R) \) is completely semiprime. Let \( a \in R \). Since \( R \) is a weak \((\sigma, \delta)\)-rigid ring. Let \( \{a(\sigma(a) + \delta(a))\}^2 \in N(R) \). Hence there exists a positive integer \( n \) such that \( [a^2(\sigma(a) + \delta(a))]^n = 0 \) which implies that \( a^{2n}(\sigma(a) + \delta(a))^{2n} = 0 \). But by Proposition (2.1.19), \( \sigma(a) + \delta(a) \neq 0 \). Hence \( a^{2n} = 0 \) which implies that \( a^n = 0 \), because \( R \) is an integral domain. Therefore, \( a^n(\sigma(a) + \delta(a))^n = 0 \) or \( \{a(\sigma(a) + \delta(a))\}^n = 0 \). Thus \( a(\sigma(a) + \delta(a)) \in N(R) \). Hence \( N(R) \) is completely semiprime. \( \square \)
The converse of Theorem (4.3.2) is not true as seen in example (4.3.3).

**Example 4.3.3:** Let $F$ be a field. Let $R = F \times F$ and $\sigma$ an automorphism of $R$ defined by

$$\sigma((a, b)) = (b, a) \text{ for } a, b \in F.$$ 

Then $R$ is a reduced ring and so $N(R) = \{0\}$ and therefore, it is completely semiprime. Let $r \in F$. Define $\delta_r : R \to R$ by

$$\delta_r((a, b)) = (a, b)r - r\sigma((a, b)) \text{ for } a, b \in F.$$ 

Clearly, $\delta_r$ is a $\sigma$-derivation of $R$. Also $R$ is not a weak $(\sigma, \delta)$-rigid ring. For take $(1, -1) \in R$, $r = \frac{1}{2}$. Then

$$\delta_r((1, -1)) = (1, -1)\frac{1}{2} - \frac{1}{2}\sigma((1, -1)) = (1, -1)$$

and

$$\delta_r((1, -1)) = (1, -1)[\sigma((1, -1)) + \delta_r((1, -1))] = (1, -1)[(1, 1) + (1, -1)] = (1, -1)(0, 0) = (0, 0) \in N(R).$$

But $(1, -1) \notin N(R)$.

Using the above Theorem (4.3.2), it follows that

**Corollary 4.3.4:** Let $R$ be a commutative Noetherian integral domain which is also an algebra over $\mathbb{Q}$. Let $\sigma$ be an automorphism of $R$ and $\delta$ a $\sigma$-derivation of $R$. Then $R$ is a weak $(\sigma, \delta)$-rigid ring implies that $N(R)$ is completely semiprime.