FIXED POINTS FOR A CLASS OF OPERATORS OVER DIFFERENT TOPOLOGICAL STRUCTURED SPACES

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Statement

The work submitted in this Ph.D Thesis has been carried out by the candidate under the supervision of Professor Mantu Saha of the Department of Mathematics, The University of Burdwan, Burdwan, West Bengal, India.

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I am to certify that Amit Kumar Laha, Senior Research Fellow (UGC) has worked for his Ph.D. thesis entitled "FIXED POINTS FOR A CLASS OF OPERATORS OVER DIFFERENT TOPOLOGICAL STRUCTURED SPACES" under my supervision, and that he has fulfilled all the requirements of the University regulations relating to course work, residential requirement, publications in journals having ISSN number and presentation of pre-submission seminar talks. It is also certified that this thesis incorporates the results of original investigations made by Amit Kumar Laha under my continuous guidance, and that work contained in this thesis is, in my opinion, worth considering for the award of Ph.D. Degree. I also certify that the said thesis now submitted is in partial fulfilment of Ph.D. degree in Mathematics of The University of Burdwan, and has not been submitted previously anywhere for any degree either by candidate or by anyone else.

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Contents

Chapter 1

Introduction

The study of fixed point of mappings in different topological spaces is an interesting study for sake of its own and it has become useful tool because the study finds applications in different branches of Mathematics like ordinary differential equations and linear algebraic equations, etc.

In the study of literatures one can see that *Henri Poincaré* ([61]) is the pioneer in the theory of fixed point analysis. The most celebrated work in study of fixed point of mappings goes back in 1922 due to Polish Mathematician, Stefan Banach ([4]) by whose name we are now familiar with "Banach contraction Principle" in a complete metric space. Researchers in this area recognized this principle as a fundamental one.

After this, researchers had been tempted to apply Fixed point theorems in many important areas as diverse as differential equations, random equations, optimal control problem, approximate theory, operation research, mathematical economics, etc. and they had successfully extended Banach Contraction Principle in many directions. On examinations of these useful applications of fixed point theorem and on scrutiny of extensions of Banach Contraction Principle, it is not difficult to conclude that extensions had been accomplished

(i) when underlying spaces had been made more relaxed,

(ii) nature of the mappings acting on the space has been assumed to be less restrictive, and

(iii) a combinations of both (i) and (ii).

While type of mappings dealt with in fixed point theory has undergone sea changes, say from being contraction mapping to a discontinuous one, it has now been established that the contractive character of some of the iterates of mappings had been responsible for assigning a fixed point of mapping; on the other hand, underlying spaces had been taken more general than metric space (X, d).

In 1962, Edelstein ([26]), in his paper "On fixed points and periodic points under contractive mappings" appeared in Journal of London Mathematical Society, had considered a class of mappings as contractive mapping T that satisfies

$$d(Tx, Ty) < d(x, y)$$

for $x, y \in X$ where $x \neq y$ and he had proved a fixed point theorem for such mapping. The contractive mapping is strictly larger than that of contraction mapping. A contractive mapping of a complete metric space into itself need not have a fixed point.

For example, let $X = \{x \in \mathbb{R} : x \geq 1\}$ with the usual metric $d(x, y) = |x - y|$ where $\mathbb{R}$ be the set of reals. Let $T : X \to X$ given by $T(x) = x + \frac{1}{x}$.

It is to be noted that, the iterative sequence $\{T^n x\}$ for any $x \in X$ here, is not good enough to produce a fixed point like a contraction mapping T converges to a fixed point in X.

As seen from the example just given above that T is contractive but has no fixed point. Also note that T is not a contraction. In 1969, Belluce and Krik ([6]) have considered a class of mappings called non-expansive mappings T over (X, d) satisfying

$$d(Tx, Ty) \leq d(x, y)$$

for all $x, y \in X$.

It is clear from these mappings that contraction $\Rightarrow$ contractive $\Rightarrow$ non-expansive. Only common feature in this chain is that the ratio of image distances and pre-image distance is bounded by a positive constant and consequently mappings are necessarily continuous.

On the other hand in 1968, Kannan ([46]) focussed on this for the first time and proved a fixed point theorem for a class of discontinuous mappings, we call them Kannan mapping T acting on (X, d) into itself and satisfying

$$d(T(x), T(y)) \leq \beta[d(x, T(x)) + d(y, T(y))]$$

for all $x, y \in X$, with $0 \leq \beta < \frac{1}{2}$.
For example, let $X = [0, 1]$ with usual metric $d(x, y) = |x - y|$. Let $T : X \to X$ be given by

$$T(x) = \begin{cases} \frac{x}{4}, & \text{for} \quad x \in [0, \frac{1}{2}), \\ \frac{x}{5}, & \text{for } x \in [\frac{1}{2}, 1]. \end{cases} \tag{1.0.1}$$

Here T is discontinuous at $x = \frac{1}{2}$. But it is easily seen that T is a Kannan mapping for $\beta = \frac{4}{9}$. It is to be noted that a Kannan mapping may not be continuous but iterates of T diminish the distance between a pair of points of (X, d). Here T has a unique fixed point 0.
In continuation of the textual matters, we now exhibit the major developments of metric fixed point theory and for this we now recall some basic preliminaries along with some important results for single valued mapping on a metric space.

Definition 1.0.1. ([43]) Let X be any nonempty set and f be a map of X, or a subset of X, into X. A point $x \in X$ is called a fixed point for f if $x = f(x)$. The set of all fixed points of f is denoted by $Fix(f)$.

The typical form for finding a solution of the equation $P(z) = 0$, P is a complex polynomial, is equivalent to find fixed point of the self-map $z \to z - P(z)$ of $\mathbb{C}$. We can say more generally that, if D is any operator acting on a subset of a linear space, then the solution of the equation $Du = 0$ is equivalent to that the map $u \to u - Du$ has a fixed point.

Definition 1.0.2. ([32]) A topological space X is said to possess the fixed point property if every continuous function of X into itself has a fixed point.

Definition 1.0.3. ([32]) Let X is called a fixed point space provided every continuous map $f : X \to X$ has a fixed point.

Example 1.0.4. ([32]) Any closed bounded interval $I = [a, b] \subset \mathbb{R}$ is a fixed point space. Indeed, given a continuous function $f : I \to I$, we have $a - f(a) \leq 0$ and $b - f(b) \geq 0$; then by Intermediate Value Theorem, we can find a solution of the equation $x - f(x) = 0$ in $I = [a, b]$ i.e., f has a fixed point in $\mathbb{R}$.

Example 1.0.5. ([32]) The real line space $\mathbb{R}$ is not in general a fixed point space due to the fact that a translation function though continuous $x \to x + 1$ has no fixed point.

The most notable results in fixed point theory are as follows:

1. Schauder Fixed Point Theorem,
2. Brouwer's Fixed Point Theorem,
3. Banach's Fixed Point Theorem.

1.1 Schauder Fixed Point Theorem

Theorem 1.1.1. *([71]) Let P be a nonempty, closed, bounded and convex subset of the Banach space E, and assume that $G : P \to P$ is compact. Then G has a fixed point.*

Equivalently, Schauder Fixed Point Theorem states that: **Every convex set in a normed linear space is a fixed point space for compact maps.**

1.2 Brouwer Fixed Point Theorem

Theorem 1.2.1. *([20]) If P is a closed, bounded and convex subset of $\mathbb{R}^n$, then every continuous function $G : P \to P$ has a fixed point.*

Equivalent statement of Brouwer Fixed Point Theorem asserts that: **The closed unit ball B_1 of $\mathbb{R}^n$ has the fixed point property.**

1.3 Banach Contraction Principle and its extensions

The most significant theorem was proved by Banach ([4]) in 1922 which is known as "Banach Contraction Principle".

Theorem 1.3.1. *([43]) Let (X, d) be a complete metric space, $c \in (0, 1)$ and $f : X \to X$ be a mapping such that for each $x, y \in X$,*

$$d(fx, fy) \leq cd(x, y) \tag{1.3.1}$$

then f has a unique fixed point $a \in X$, such that for each $x \in X$, $\lim_{n\to\infty} f^n x = a$.

Equivalently, Banach Contraction Principle asserts that: **Every complete metric space is a fixed point space for contraction maps.**

By Weakening the contractivity condition of the mapping, lots of generalizations of Banach Contraction Principle on an arbitrary complete metric space have been proved.

Theorem 1.3.2. *([32]) Let (X, d) be a complete metric space and $F : X \to X$ a map, not necessarily continuous. Assume for each $\epsilon > 0$ there is a $\delta(\epsilon) > 0$ such that if $d(x, Fx) < \delta$, then $F[B(x, \epsilon)] \subset B(x, \epsilon)$. Then, if $d(F^n u, F^{n+1}u) \to 0$ for some $u \in X$, the sequence $\{F^n u\}$ converges to a fixed point for F.*

Theorem 1.3.3. *([32]) Let (X, d) be complete, and let $F : X \to X$ be a map satisfying*

$$d(Fx, Fy) \leq \varphi[d(x, y)],$$

where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is any nondecreasing (not necessarily continuous) function such that $\varphi^n(t) \to 0$ for each fixed $t > 0$. Then F has a unique fixed point u, and $F^n x \to u$ for each $x \in X$.

Theorem 1.3.4. *([19]) Let F be a mapping from a complete metric space X into itself. Suppose there exists a function ϕ upper semi continuous from right from $\mathbb{R}^+$ into itself such that*

$$d(Fx, Fy) \leq \phi[d(x, y)]$$

for all $x, y \in X$. If $\phi(t) < t$ for each $t > 0$. Then F has a unique fixed point in X and for every x in X, $\lim_{n\to\infty} F^n x = u$.

Theorem 1.3.5. *([32]) Let (X, d) be complete, and let $F : X \to X$ be a map satisfying*

$$d(Fx, Fy) \leq \alpha(x, y)d(x, y),$$

where $\alpha : X \times X \to \mathbb{R}^+$ has the property: for any closed interval $[a, b] \subset \mathbb{R}^+ - \{0\}$,

$$\sup\{\alpha(x, y)|a \leq d(x, y) \leq b\} = \lambda(a, b) < 1.$$

Then F has a unique fixed point u, and $F^n x \to u$ for each $x \in X$.

Various results on fixed points of contractive mapping and their generalizations have been surveyed by Rhoades (see- Rhoades([66])).
Also we can recall a mixed type mapping due to Reich ([65]) that satisfies

$$d(Tx, Ty) \leq \alpha_1 d(x, Tx) + \alpha_2 d(y, Ty) + \alpha_3 d(x, y)$$

where $\alpha_1, \alpha_2, \alpha_3 \geq 0$ with $\alpha_1 + \alpha_2 + \alpha_3 < 1$, for all $x, y \in X$.
This class of self mappings is takes care both Kannan type mapping and contraction mapping as well in its fold. The inequality $\alpha_1 + \alpha_2 + \alpha_3 < 1$ plays the key role in proving fixed point theorems for such mappings. Researchers continued to deal with the class of mappings with their iterates enjoying contractive nature and satisfying conditions more relaxed than, though similar to above as stated to make the class still broader. The first result for families of mappings was proved by Markov ([52]) in 1936. Kakutani ([44]) gave a direct proof of Markov's theorem in 1938 and also proved a fixed point theorem for affine equi-continuous mappings. A relevant question may be posed: Does the convergence of a sequence of mappings $\{T_i\}$ in a metric space to a mapping T implies the convergence of the sequence of their fixed points to a fixed point of T?
It was answered in the following developments where we find two types of convergence of mappings: (i) pointwise convergence and (ii) uniform convergence. The first theorem regarding the continuity of fixed points of contraction mapping was proved by British Mathematician Bonsall ([16]). Subsequently Nadler Jr.([57])

obtained results on converging sequence of contraction mappings and also gave an application of it.

1.4 Multivalued mapping

We now discuss on the theory of fixed points for a multivalued contraction mappings or generalized multivalued mappings on a metric space. The study of fixed point problem on such mappings was initiated by Kakutani ([45]) in 1941 in finite dimensional spaces. It was then extended to infinite dimensional Banach spaces by Bohnenblust and Karlin ([15]) in 1950 and to locally convex spaces by Fan ([28]) in 1952.

Fixed point theorem for multifunctions provide natural setting for many problems in control theory (See-[25]) involving differential equations. Also they have been effectively used in mathematical problems of Game theory and various mathematical models in Economics. We shall now state relevant results in this direction.

The developments of geometric fixed point theory for multifunction was initiated by Nadler Jr.([57]), subsequently pursued by Markin (([50])-([51])), Assad and Kirk ([3]), Browder ([20]), Goebel ([31]), Reich ([64]) and others. Since then fixed point theorems for multifunction have been extensively studied by many researchers. We shall take some basic definitions and results as preliminaries.

Given a metric space (X, d), we denote the collection of all non empty subsets of X, class of all non empty bounded subsets of X, class of all nonempty closed subsets and class of all compact subsets of X by $P(X)$, $B(X)$, $Cl(X)$ and $\mathcal{H}_c(X)$ respectively. Let $CB(X)$ denotes the class of all closed bounded subsets of X.

For $A, B \in CB(X)$ defined a metric called Hausdorff metric by

$$H(A, B) = \max\{\sup_{p \in A} d(p, B), \sup_{q \in B} d(A, q)\}.$$

The metric H depends on the metric on d of X and two equivalent metric on X may not generate equivalent Hausdorff metrics for $CB(X)$ (see-[48]).

Let (X, d) be a metric space and let $F : X \to CB(X)$ be a multivalued mapping.

Definition 1.4.1. ([21]) A mapping $F : X \to CB(X)$ is said to be a multivalued contraction if there exists a constant $0 \leq \alpha < 1$ such that

$$H(Fx, Fy) \leq \alpha d(x, y)$$

for all $x, y \in X$.

Example 1.4.2. Let $X = [0, 1]$ and $f : X \to X$ be such that

$$f(x) = \begin{cases} \frac{1}{2}x + \frac{1}{2}, & \text{for } \ 0 \leq x \leq \frac{1}{2}, \\ -\frac{1}{2}x + 1, & \text{for } \ \ 0 \leq x \leq \frac{1}{2}. \end{cases} \tag{1.4.1}$$

Define $F : X \to 2^X$ by $F = \{0\} \cup \{f(x)\}$ for each $x \in X$. Then one can check that F is a multivalued contraction mapping and the set of fixed point of F is $(0, \frac{2}{3})$.

Definition 1.4.3. ([21]) A point $x \in X$ is said to be a fixed point of the multivalued mapping $F : X \to CB(X)$ (single valued mapping f) if $x \in F(x)(x = fx)$. A point $x \in X$ is said to be a coincidence point of F and f if $f(x) \in Fx$.
Let $F : X \to P(X \times X)$ be a multivalued mapping. A point $(x, y) \in X \times X$ is said to be a coupled fixed point of the multivalued mapping F if $x \in F(x, y)$ and $y \in F(y, x)$.

Definition 1.4.4. [57] A multivalued function $F : X \to CB(X)$ is said to be contractive if and only if for each $x_1, x_2 \in X$ with $x_1 \neq x_2$

$$H(Fx_1, Fx_2) < d(x_1, x_2)$$

Definition 1.4.5. ([43]) A multivalued function $F : X \to CB(X)$ is said to be non-expansive if and only if for each $x_1, x_2 \in X$ with $x_1 \neq x_2$

$$H(Fx_1, Fx_2) \leq d(x_1, x_2)$$

We now refer a family of mappings which satisfies generalized Kannan-Reich condition and also cite a result for the existence of common fixed point of a pair of such multivalued mappings. The result given below is due to Bose and Mukherjee ([17]).

Theorem 1.4.6. *Let F_1 and F_2 be two self mappings over a complete metric space X into $CB(X)$ satisfying the following conditions*

$$H(F_1x, F_2y) \leq a_1 D(x, F_1x) + a_2 D(y, F_2y)$$
$$+a_3 D(y, F_1x) + a_4 D(x, F_2y) + a_5 d(x, y)$$

where $D(x, A)$ is the distance between x and a subset A of X, for all $x, y \in X$ where $a_i \geq 0$ for $i = 1, 2, 3, 4, 5$ with $\sum_{i=1}^{5} a_i < 1$ and $a_1 = a_2$ or $a_3 = a_4$. Then F_1 and F_2 have a common fixed point in X.

The following result was also proved by Bose and Mukherjee ([17]) which is a generalization of a theorem of Iseki ([40]).

Corollary 1.4.7. *Let (X, d) be a complete bounded metric space and let $F : X \to CL(X)$ be a multivalued mapping satisfying the following condition:*

$$H(Fx, Fy) \leq a_1 D(x, Fx) + a_2 D(y, Fy)$$
$$+a_3 D(y, Fx) + a_4 D(x, Fy) + a_5 d(x, y)$$

for any $x, y \in X$ where $a_i \geq 0$ for $i = 1, 2, 3, 4, 5$ with $\sum_{i=1}^{5} a_i < 1$, $a_1 = a_2$ and $a_3 = a_4$. Then F has a fixed point in X.

1.5 Applications of fixed point theory

In the study of non linear phenomena, the theory of fixed points plays a pivotal role in solving different types of problems. Applying the techniques as available in fixed point theory we are now able to solve and analyze the nature of the problems.

A fruitful applications could be found in control theory, category theory, functional equations, mathematical economics and many other areas. Theory of fixed point plays a key role in analyzing the existence of solutions in the theory of random differential equations, numerical methods etc. In recent past years, fixed point theory has a major role in the theory of algebra (homotopic fixed point), the theory of automata, linear functional analysis and in the theory of critical points, etc.

1.6 Discussion on Chapters

We now put our contributions in finding results of fixed points of mappings over different topological structured spaces in different chapters of this Ph.D. thesis.

1.6.1 Chapter II

This chapter deals with the study on approximate fixed point which is inevitable consequences in fixed point theory. For the sake of interest in mathematical analysis related to application oriented problems the study of fixed point theory is very similar to a study of approximate fixed point theory. If T be a self map of a metric space (X, d), then a point z is called an approximate solution of the equation $Tx = x$, or equivalently, $z \in X$ is an approximate fixed point (ϵ -fixed point) of T if $d(Tz, z) \leq \epsilon$ satisfies, where ϵ is a positive number. In practical situations an approximate solution and approximate fixed point plays a useful role when a mapping under consideration may not have a fixed point due to conditions imposed in problems as we encounter.

In reality, approximate solution is more than sufficient where the existence of fixed point may not be assured. Recently, approximate fixed point theory finds applications in mathematical economics, non-cooperative game theory, dynamic programming, nonlinear analysis, variational calculus, theory of integro-differential equations and several other areas of mathematical analysis.

Motivated by the article of Tijs, Torre and Brănzei ([79]) and M. Berinde ([8]) some results on approximate fixed point theorems in metric space have been proved in this chapter.

We now recall some basic definitions and results:

Definition 1.6.1. ([60]) Let $f : X \to X$, $\epsilon > 0$, $x_0 \in X$. An element x_0 is called an ϵ-fixed point (or approximate fixed point) of f provided that

$$d(f(x_0), x_0) < \epsilon$$

For a given $\epsilon > 0$, we will denote the set of all ϵ-fixed points of f by ([60]):

$$Fix_\epsilon(f) = x \in X : x \ is \ an \ \epsilon - \ fixed \ point \ of \ f.$$

Definition 1.6.2. ([60]) Let $f : X \to X$. Then f has the approximate fixed point property if, $\forall \epsilon > 0$, $Fix_\epsilon(f) \neq \phi$.

Definition 1.6.3. ([60]) A mapping $f : X \to X$ is called weak contraction if there exist $\alpha \in (0,1)$ and $L \geq 0$ such that

$$d(f(x), f(y)) \leq \alpha d(x,y) + Ld(y, f(x)),$$

for all $x, y \in X$.

Lemma 1.6.4. *([60]) Let (X,d) be a metric space, $f : X \to X$ such that f is asymptotically regular, i.e. $d(f^n(x), f^{n+1}(x)) \to 0$ as $n \to \infty$, for all $x \in X$. Then f has the approximate fixed point property.*

In the following, by $\delta(A)$ for a set $A \neq \phi$ of X, we will understand the diameter of the set A, i.e. $\delta(A) = \{\sup d(x,y) | x, y \in A\}$.

Lemma 1.6.5. *([60]) Suppose (X,d) be a metric space and let $f : X \to X$ be a mapping. Let $\epsilon > 0$. We assume that:*
i) f has the approximate fixed point property;
ii) for each $\eta > 0$, there exists $\varphi(\eta) > 0$ such that

$$d(x,y) - d(f(x), f(y)) \leq \eta \Rightarrow d(x,y) \leq \varphi(\eta),$$

for all $x, y \in Fix_\epsilon(f)$. Then $\delta(Fix_\epsilon(f)) \leq \varphi(2\epsilon)$.

Theorem 1.6.6. *([60]) Let (X,d) be a metric space and $f : X \to X$, a weak contraction mapping. Then f has the approximate fixed point property.*

Definition 1.6.7. ([60]) Let (X,d) be a metric space and $f : X \to X$. We say that f is a Rus-Reich operator if there exist $a, b, c \in \mathbb{R}^+$ with $a + b + c < 1$ such that

$$d(f(x), f(y)) \leq ad(x,y) + bd(x, f(x)) + cd(y, f(y)),$$

for all $x, y \in X$.

Theorem 1.6.8. *([60]) Let (X, d) be a metric space and $f : X \to X$ a Rus-Reich type operator. Then f has the approximate fixed point property.*

In this chapter our main aim is to study the approximate coincidence point of two nonlinear mappings due to Geraghty ([30]) in 1973 and Mizoguchi and Takahashi ([53]) in 1989.

By weakening the condition and with withdrawal of completeness property in a underlying space, the existence of ϵ-fixed points for such mappings have been proved.

1.6.2 Chapter III

Cone metric space is a generalization of metric space where every pair of elements is assigned to an element of an ordered Banach space. Assuming vector-valued metrics with values in an ordered real Banach space, Huang and Zhang ([39]) first introduced the concept of cone metric space with suitable definitions. They were also able to prove some fixed point theorems of different types of contractive mappings in cone metric spaces. Many topological questions arose in cone metric spaces out of which cone metric space was shown as a first countable topological space by ([80]). It was also proved that in cone metric space, continuity is equivalent to sequential continuity and compactness is equivalent to sequential compactness.

We now recall some basic definitions and results:

Definition 1.6.9. [67] Let E be real Banach space and P, a subset of E. Then P is said to be a cone whenever

(i) P is closed, non-empty and $P \neq \{0\}$,

(ii) $ax + by \in P$ for all $x, y \in P$ and for non-negative real numbers a, b ,

(iii) $P \cap (-P) = \{0\}$.

For a given cone $P \subseteq E$, we can define the partial ordering $' \leq '$ with respect to P by $a \leq b$ if and only if $b - a \in P$, $a < b$ stands for $a \leq b$ but $a \neq b$ and while $a \ll b$ stands for $b - a \in IntP$, where $IntP$ denotes the interior of P.

Definition 1.6.10. [67] Let X be a non-empty set. Suppose the mapping $d : X \times X \to E$ satisfies

(i) $0 \leq d(x,y), \forall x,y \in X$ and $d(x,y) = 0$ if and only if $x = y$,

(ii) $d(x,y) = d(y,x)$, $\forall x,y \in X$,

(iii) $d(x,y) \leq d(x,z) + d(z,y)$, $\forall x,y,z \in X$.

Then we say that d is a cone metric on X and (X,d) is a cone metric space.

Definition 1.6.11. [67] Let (X,d) be a cone metric space, $x \in X$ and $\{x_n\}$ be a sequence in X. Then

(A) $\{x_n\}$ converges to $x \in X$ whenever for every $c \in E$ with $0 \ll c$ there is a natural number N such that $d(x_n, x) \ll c$ for all $n \geq N$. We denote it by $\lim\limits_{n\to\infty} x_n = x$ or $x_n \to x (n \to \infty)$.

(B) $\{x_n\}$ is a Cauchy sequence whenever for every $c \in E$ with $0 \ll c$ there is a natural number N such that $d(x_n, x_m) \ll c$ for all $n,m \geq N$ or $d(x_n, x_m) \to 0 (n,m \to \infty)$.

(C) (X,d) is a complete cone metric space if every Cauchy sequence in X is convergent in X.

The cone P is said to be a normal cone if there exists a real number $k > 0$ such that $\forall x,y \in E$, $0 \leq x \leq y \Rightarrow \|x\| \leq k\|y\|$. The least positive number k satisfying the above is called the normal constant of P. Clearly $k \geq 1$.
In the following we always suppose that E is a normed space and P is a cone in E with normal constant $k = 1$, $IntP \neq \phi$ and " $\leq$ " is partial ordering with respect to P.

Lemma 1.6.12. *[39] Let (X,d) be a cone metric space, P be a normal cone with normal constant k. Let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $d(x_n, x) \to 0 (n \to \infty)$.*

Theorem 1.6.13. *[39] Let (X,d) be a complete cone metric space, P be a normal cone with normal constant K. Suppose the mapping $T : X \to X$ is a contraction mapping; that is*

$$d(Tx, Ty) \leq k d(x,y),$$

for all $x, y \in X$, where $k \in [0,1)$ is a constant. Then T has a unique fixed point in X. And for any $x \in X$, iterative sequence $\{T^n x\}$ converges to the fixed point.

Theorem 1.6.14. *[39] Let (X,d) be a complete cone metric space, P be a normal cone with normal constant K. Suppose the mapping $T : X \to X$ is a Kannan mapping, that is*

$$d(Tx,Ty) \leq k[d(Tx,y) + d(x,Ty),$$

for all $x, y \in X$, where $k \in [0,1/2)$ is a constant. Then T has a unique fixed point in X. And for any $x \in X$, iterative sequence $\{T^n x\}$ converges to the fixed point.

After that many researchers had proved several fixed point theorems for different types of single valued contractive mappings and multivalued contractive mappings over such cone metric space.

Lemma 1.6.15. *[67] Let (X,d) be a cone metric space, P be a normal cone with normal constant equal to 1, and A be a compact set in (X,τ_c). Then for every $x \in X$, $\exists\ a_0 \in A$ such that $\|d(x,a_0)\| = \inf_{a\in A} \|d(x,a)\|$.*

Lemma 1.6.16. *[67] Let (X,d) be a cone metric space, P be a normal cone with normal constant equal to 1, and A, B are two compact sets in (X,τ_c).
Then, $\sup_{x\in B} d^/(x,A) < \infty$, where, $d^/(x,A) = \inf_{a\in A} \|d(x,a)\|$.*

Definition 1.6.17. [67] Let (X,d) be a cone metric space, P be a normal cone with normal constant equal to 1, $\mathcal{H}_c(X)$ be the set of all compact subsets of (X,τ_c) and $A \in \mathcal{H}_c(X)$. then we define

$$h_A : \mathcal{H}_c(X) \to [0,\infty) \text{ by } h_A(B) = \sup_{x\in A} d^/(x,B). \tag{1.6.1}$$

and

$$d_H : \mathcal{H}_c(X) \times \mathcal{H}_c(X) \to [0,\infty) \text{ by } d_H(A,B) = \max\{h_A(B), h_B(A)\}. \tag{1.6.2}$$

Remark 1.6.18. [67] Let (X,d) be a cone metric space with normal constant equal to 1. Define $\rho : X \times X \to [0,\infty)$ by $\rho(x,y) = \|d(x,y)\|$. Then (X,ρ) is a metric space.

Remark 1.6.19. [67] For each $A, B \in \mathcal{H}_c(X)$ and $x, y \in X$, we have the following relations

(a) $d^{/}(x,A) \leq \|d(x,y)\| + d^{/}(y,A)$.

(b) $d^{/}(x,A) \leq d^{/}(x,B) + h_B(A)$.

(c) $d^{/}(x,A) \leq \|d(x,y)\| + d^{/}(y,B) + h_B(A)$.

Theorem 1.6.20. *[67] Let (X,d) be a complete cone metric space with normal constant $K = 1$, and the multifunction $T : X \to \mathcal{H}_c(X)$ satisfies the condition*

$$d_H(Tx,Ty) \leq c(d^{/}(Tx,x) + d^{/}(Ty,y))$$

for all $x,y \in X$, where $c \in (0,1/2)$ is a constant. Then T has a fixed point in X.

Theorem 1.6.21. *[67] Let (X,d) be a complete cone metric space with normal constant $K = 1$, and the multifunction $T : X \to \mathcal{H}_c(X)$ satisfies*

$$d_H(Tx,Ty) \leq c(d^{/}(Tx,y) + d^{/}(Ty,x))$$

for all $x,y \in X$, where $c \in (0,1/2)$ is a constant. Then T has a fixed point in X.

In this chapter we use the notation of $\alpha - \psi$ multi-valued contractive mapping for finding a fixed point in the setting of cone metric space with normal constant equal to 1.

1.6.3 Chapter IV

In ordered metric spaces existence of fixed points was established by Ran and Reurings ([63]). Replacing the order structure with a graph structure on a metric space, Jachymski ([42]) first introduced a new way in searching of fixed point of an mapping on a metric space endowed with a graph. After that, $Gw\acute{o}\acute{z}d\acute{z} - Lukawska$ and Jachymski ([33]) developed the theory for fixed points of mappings on a metric space endowed with a directed graph due to Hutchinson-Barnsley ([34]). Subsequently, Abbas ([1]) proved some fixed point theorems for a power graph contraction mapping. In 2010, Bojor ([13]) initiated a clear concept of fixed point of contraction mapping on a metric space endowed with a graph. Recently, Bojor ([13]) proved some fixed point theorems for different types of contractive mappings on a graph structured metric space. The study of fixed point theory for a single valued and multi-valued operators over such metric spaces was done by Beg et.al. ([5])

A graph G is said to be a directed graph if the ordered pair $(V(G), E(G))$ where $V(G)$ set of vertices and $E(G)$ set of edges, disjoint from $V(G)$ together with an incidence function G that associates with each edge of G, an ordered pair of (not necessarily distinct) vertices of $V(G)$.

By G^{-1} we mean the graph obtained from G by reversing the direction of edges where

$$E(G^{-1}) = \{(x, y) \in X \times X : (y, x) \in E(G)\}. \tag{1.6.3}$$

and $V(G^{-1}) = V(G)$ and denote $\tilde{G}$ as a directed graph for which the set of its edges is symmetric and defined by

$$E(\tilde{G}) = E(G^{-1}) \cup E(G) \tag{1.6.4}$$

and $V(\tilde{G}) = V(G)$.

A graph G is connected if there is a path between any two vertices and G is said to be weakly connected if $\tilde{G}$ is connected.

We now recall some basic definitions and results:

Definition 1.6.22. ([13]) A mapping $F : X \to X$ is said to be a Banach G-contraction or simply G-contraction if f preserves edges of G, i.e.,

$$\forall x, y \in X, (x, y) \in E(G) \Rightarrow (fx, fy) \in E(G),$$

and f decreases weights of edges of G in the following way:

$$\exists \alpha \in (0, 1), \forall x, y \in X, (x, y) \in E(G) \Rightarrow d(fx, fy) \leq \alpha d(x, y).$$

Theorem 1.6.23. *([13]) The following statements are equivalent:*
(i) G is weakly connected;
(ii) for any G-contraction $f : X \to X$ and given $x, y \in X$, the sequences $(f^n x)_{n\in\mathbb{N}}$ and $(f^m x)_{n\in\mathbb{N}}$ are Cauchy equivalent;
(iii) for any G-contraction $f : X \to X$, $card(Fix(f)) \leq 1$.

Theorem 1.6.24. *([13]) Let (X, d) be complete. The following statements are equivalent:*
(i) G is weakly connected;
(ii) for any G-contraction $f : X \to X$, there is $x^* \in X$ such that $\lim_{n\to\infty} f^n x = x^*$, for all $x \in X$.

Definition 1.6.25. ([13]) Let (X, d) be a metric space and G a graph. The mapping $T : X \to X$ is said to be a G-Kannan mapping if:

$$\forall x, y \in X, (x, y) \in E(G) \Rightarrow (Tx, Ty) \in E(G),$$

and there exists $a \in [0, \frac{1}{2})$ such that:

$$d(Tx, Ty) \leq a[d(x, Tx) + d(y, Ty)]$$

for all $(x, y) \in E(G)$.

The Pompeiu-Hausdorff ([24]) functional

$$H : P(X) \times P(X) \to \mathbb{R}^+ \cup \{\infty\},$$

is defined by

$$H(A, B) = \max\{\sup_{a\in A} D(a, B), \sup_{b\in B} D(A, b)\}.$$

The diameter generalized functional δ generated by d is given by

$$\delta : P(X) \times P(X) \to \mathbb{R}^+ \cup \{\infty\},$$

and it is defined by

$$\delta(A, B) = \sup\{d(a, b) : a \in A, b \in B\}.$$

In a metric space (X, d) and let $T : X \to P(X)$ be a multivalued operator, then $x \in X$ is said to a fixed point for T iff $x \in Tx$ and the set $Fix(T) = \{x \in X : x \in Tx\}$ is called fixed point set of T and $SFix(T) = \{x \in X : \{x\} = Tx\}$ is called strict fixed point set of T.
Let the graph of T is denoted by $G(T)$ and we write $G(T) = \{(x, y) : y \in Tx\}$.

Theorem 1.6.26. *([24]) Let (X, d) be a complete metric space and let G be a directed graph such that the triple (X, d,G) satisfies the following property:*
for any sequence $\{x_n\}_{n\in\mathbb{N}} \in X$ *with* $\{x_n\} \to x \in X$ *as* $n \to \infty$*, there exists a subsequence* $\{x_{k_n}\}_{k\in\mathbb{N}}$ *of* $\{x_n\}$ *satisfying* $(x_{k_n}, x) \in E(G)$*.*
Let $T : X \to P_b(X)$ *be a multivalued operator. Suppose that the following assertions hold:*

(i) There exists $a, b, c \in \mathbb{R}_+$ *with* $b \neq 0$ *and* $a + b + c < 1$ *such that*

$$\delta(Tx, Ty) \leq ad(x, y) + b\delta(x, Tx) + c\delta(y, Ty)\},\quad \text{for all } (x, y) \in E(G). \tag{1.6.5}$$

(ii) For each $x \in X$ *the set*

$$\tilde{X}_T(x) = \{y \in Tx : (x, y) \in E(G) \text{ and } \delta(x, Tx) \leq qd(x, y), \text{for some } q \in (1, \tfrac{1-a-c}{b})\} \text{ is nonempty.} \tag{1.6.6}$$

Then we have,

(I) $Fix(T) = SFix(T) \neq \Phi$

(II) If we further assume that $x^*, y^* \in Fix(T) \Rightarrow (x^*, y^*) \in E(G)$ *then* $Fix(T) = SFix(T) = \{x^*\}$*.*

Some fixed point theorems for a class of generalized contractive mappings by employing multi-valued Geraghty's mappings on a metric space endowed with a graph have been proved in this chapter.

1.6.4 Chapter V

Transferring the contractive property of the nonlinear mappings into the monotonicity property in metrical fixed point theory, we consider a partial ordered metric space $(X, d, \leq)$ which was initiated by Ran and Reurings ([63]) and followed by Nieto and Rodriguez-Lopez ([58]), Wang ([82]). Using this concept of monotonicity property in a classical sense, existence of common fixed points for a pair of nonlinear mappings have been proved. Coupled fixed point and some allied fixed point theorems were also proved by Bhaskar and Lakshmikantham ([9]) in 2006 under certain conditions. Motivated upon such works Lakshmikantham and *Ćirić* ([49]) had proved some generalized results by introducing concept of the mixed g-monotone property. Many researchers find interest in using this property in different spaces like probabilistic metric spaces, metric spaces endowed with a graph etc. for finding applications in solving the existence of solutions in differential equations, integral equations, matrix equations etc.

Let X be a non-empty set and we denote $X \times X ... \times X(k\ times)$ by X^k where $k \in \mathbb{N}$ and $\varphi^n(t)$ by the composition of the function φ for n times.

We say two elements x, y of a partial ordered set $(X, \leq)$ are comparable if $x \asymp y$ (i.e. $x \leq y$ or $y \leq x$.)

Let $\varphi : [0, \infty) \to [0, \infty)$ be the function satisfies

a) $\varphi(t) < t$ for all $t \in (0, \infty)$,

b) $\lim_{r \to t^+} \varphi(r) < t$ for all $t \in (0, \infty)$.

We denote the collection of all such function by Φ.

We now recall some basic definitions and results:

Definition 1.6.27. ([82]) Let $(X, \preceq)$ be a partially ordered set and d be a metric on X. We say that $(X, d, \preceq)$ is regular if the following conditions hold:

(i) if a non-decreasing sequence $\{x_n\}$ is such that $x_n \to x$, then $x_n \preceq x$ for all n,

(ii) if a non-increasing sequence $\{y_n\}$ is such that $y_n \to y$, then $y_n \succeq y$ for all n.

Let $(y_1, y_2, ..., y_k), (v_1, v_2, ..., v_k) \in X^k$. The natural partial ordering in the product space X^k is

$$(y_1, y_2, ..., y_k) \preceq (v_1, v_2, ..., v_k) \Leftrightarrow y_i \preceq_i v_i$$

which will be denoted in the sequel, for convention, we write by $\preceq$, also. Clearly, $(X^k, \preceq)$ is a partially ordered set. Particularly, we denote by A the odd numbers in $\wedge_k$ and by B its even numbers. The mapping $\rho_k : X^k \times X^k \to [0,\infty)$, given by

$$\rho_k(Y,V) = \frac{1}{k}[d(y_1,v_1) + d(y_2,v_2) + ... + d(y_k,v_k)], \qquad (1.6.7)$$

where $Y = (y_1, y_2, ..., y_k), V = (v_1, v_2, ..., v_k) \in X^k$, defines a metric on X^k.
If $Y_n = (y_1^n, y_2^n, ..., y_k^n), Y = (y_1, y_2, ..., y_k) \in X^k$ then it is easy to show that

$$\rho_k(Y_n, Y) \to 0(as \quad n \to \infty) \Leftrightarrow d(y_i^n, y_i) \to 0(as \quad n \to \infty)$$

Definition 1.6.28. ([82]) Let $(X, \preceq)$ be a partially ordered set and T be a self-mapping on X^k. We say that T has an isotone property if, for any $Y_1, Y_2 \in X^k$,

$$Y_1 \preceq Y_2 \Rightarrow T(Y_1) \preceq T(Y_2)$$

Definition 1.6.29. ([82]) An element $Y \in X^k$ is called a fixed point of the mapping $T : X^k \to X^k$ if $T(Y) = Y$.

Definition 1.6.30. Two mappings $T : X^k \to X^k$ and $G : X^k \to X^k$ are said to be commutative if $TG(Y) = GT(Y)$, for all $Y \in X^k$.

Definition 1.6.31. ([83]) Let $(X, \preceq)$ be a partially ordered set and $T : X^k \to X^k$ and $G : X^k \to X^k$ be two mappings. We say that T is a G-isotone mapping if, for any $Y_1, Y_2 \in X^k$

$$G(Y_1) \preceq G(Y_2) \Rightarrow T(Y_1) \preceq T(Y_2)$$

Definition 1.6.32. ([83]) An element $Y \in X^k$ is called a coincidence point of the mappings $T : X^k \to X^k$ and $G : X^k \to X^k$ if $T(Y) = G(Y)$. Furthermore, if $T(Y) = G(Y) = Y$, then we say that Y is a common fixed point of T and G.

Definition 1.6.33. ([75]) A function $\psi : [0,\infty) \to [0,\infty)$ is called an altering distance function if the following properties are satisfied:

(1) ψ is continuous and non-decreasing.

(2) $\psi(t) = 0$ if and only if $t = 0$.

Let Ψ be the collection of all such altering distance functions ψ as defined in Definition (1.6.33).

Theorem 1.6.34. *[82] Let $(X, \leq)$ be a partially ordered set and suppose that there is a metric d on X such that (X, d) is a complete metric space. Let $T : X^k \to X^k$ be an isotone mapping for which there exist $\phi \in \Phi$ and $\psi \in \Psi$ such that, for all $Y, V \in X^k$ with $Y \geq V$,*

$$\varphi(\rho_k(T(Y), T(V))) \leq \varphi(\rho_k(Y, V)) - \psi(\rho_k(Y, V)),$$

where ρ_k is defined by (1.6.7). Suppose either
(a) T is continuous or
(b) $(X, d, \leq)$ is regular.
Then there exists $Z_0 \in X^k$ such that $Z_0 \asymp T(Z_0)$, then T has a fixed point.

Theorem 1.6.35. *[82] Let $(X, \leq)$ be a partially ordered set and suppose that there is a metric d on X such that (X, d) is a complete metric space. Let $T : X^k \to X^k$ be an isotone mapping for which there exist $\alpha \in [0, 1)$ such that, for all $Y, V \in X^k$ with $Y \geq V$,*

$$\rho_k(T(Y), T(V)) \leq \alpha(\rho_k(Y, V)),$$

where ρ_k is defined by (1.6.7). Suppose either T is continuous or $(X, \leq, d)$ is regular. If there exists $Z_0 \in X^k$ such that $Z_0 \asymp T(Z_0)$, then T has a fixed point.

Theorem 1.6.36. *[83] Let $(X, \leq)$ be a partially ordered set and suppose that there is a metric d on X such that (X, d) is a complete metric space. Let $G : X^k \to X^k$ and $T : X^k \to X^k$ be a G-isotone mapping $\phi \in \Phi$ such that, for all $Y, V \in X^k$ with $Y \geq V$ with $G(Y) \geq G(V)$,*

$$\rho_k(T(Y), T(V)) \leq \varphi(\rho_k(G(Y), G(V))),$$

where ρ_k is defined by (1.6.7). Suppose $T(X^k) \subset G(X^k)$ and also suppose either
(a) T is continuous, G is continuous and commutes with T or
(b) $(X, d, \leq)$ is regular and $G(X^k)$ is closed.
Then there exists $Y_0 \in X^k$ such that $G(Y_0) \asymp T(Y_0)$, then T and G have a coincidence point.

In this chapter we have obtained sufficient conditions for the existence of fixed point for T-isotone mappings and coincidence point for G-isotone mappings in partially ordered metric spaces.

1.6.5 Chapter VI

In this chapter, our aim has been focussed on to obtain random fixed point theorem which are random analogue of some well known deterministic fixed point theorems either available in literature or proved by us. The word "randomness" comes in probabilistic sense. In 1950's Prague school of probabilists first initiated the notion of random fixed point theory.

Almost entire work of chapter VI has been devoted to study the class of mappings $Ci(X)$ over Banach space when the individual mappings is subjected to satisfy some contractive conditions. We recall that T is a $\acute{C}iri\acute{c}$ contraction type mapping i.e. $T \in Ci(X)$ if

$$d(T^n(x), d(T^n(y) \leq q^n(x,y).\delta(x,y)$$

for all x, y belonging to a metric space (X, d), where q and δ are non-negative real valued functions over $X \times X$ not depending upon T satisfying $q(x,y) < 1$ and $\sup q(x,y) = 1$, for all $x, y \in X$. It is not difficult to see that a $\acute{C}iri\acute{c}$ type contraction mapping over a metric space may not possess a fixed point and a problem has been assigned to appropriate contractive condition over such a mapping so as to prove fixed point theorems for such $T \in Ci(X)$. In Chapter VI we have considered a deterministic mapping $T \in Ci(X)$, that satisfies

$$\begin{aligned} \|T(x) - T(y)\| \leq \quad & \alpha[\|x - T(x)\| + \|y - T(y)\|] + \beta\|x - y\| \\ & \gamma \max[\|x - T(y)\|, \|y - T(x)\|] \end{aligned} \tag{1.6.8}$$

for all $x, y \in X$ where $\alpha, \beta \geq 0$ are such that $\max\{\alpha, \beta\} + \gamma < 1$. But we have proved its random analogue in chapter VI. We have also examined the question of continuity of random fixed points of a given sequence of $\acute{C}iri\acute{c}$ ([22]) contraction type random mapping T_i which is supposed to converge to a mapping in the underlying space.

We now take some basic preliminaries (see-[43]) in continuation of literatures of random fixed point theory. Research work so far done on random fixed point theory may be examined as follows:

Špaček ([76]), Hans ([35]) proved random fixed point theorems for contraction mappings in separable Banach space. Mukherjee ([56]) gave a random version of Schauder's fixed point theorem on an atomic probability measure space while Bharucha Reid ([10]) generalized Mukherjee's result on general probability measure space. Random fixed point theorem for multivalued mappings are obtained by Itoh ([41]). Sehgal and Waters ([72]) had obtained several random fixed point theorems including random analogue of classical Rothe ([69]) fixed point theorem.
Let (X, β_X) be a separable Banach space where β_X is the σ-algebra of Borel subsets of X and let (Ω, β, μ) denotes a complete probability measure space with measure μ and β be the σ-algebra of subsets of Ω.
A mapping $x : \Omega \to X$ is said to be X-valued random variable if the inverse image of every Borel set B under mapping x belongs to β i.e. $x^{-1}(B) \in \beta$ for each $\beta \in \beta_X$.
Let Y be another Banach space. Then the mapping $F : \Omega \times X \to Y$ is said to be a random mapping if $F(\omega, x) = y(\omega)$ is a Y-valued random variable.
A mapping $F : \Omega \times X \to Y$ is said to be a continuous random mapping if the set of $\omega \in \Omega$ for what $F(\omega, x)$ is a continuous function of x has measure one.
Any mapping $x : \Omega \to X$ which satisfies the fixed point equation $F(\omega, x(\omega)) = x(\omega)$ almost surely is said to be a wide sense solution of fixed point equation and only x-valued random variable $x(\omega)$ which satisfies $\mu\{\omega : F(\omega, x(\omega)) = x(\omega) = 1\}$ is said to be a random fixed point of F.
A random fixed point of F is a wide sense solution of the fixed point equation but not conversely. The following example supports our contention.

Example 1.6.37. Let X be the set of all reals and E be a non measurable subset of X. Let $F : \Omega \times X \to X$ be a random mapping defined as

$$F(\omega, x) = x^2 + x + 1$$

for all $\omega \in \Omega$. In this case the real valued function $x(\omega)$ defined as $x(\omega) = 1$ for all $\omega \in \Omega$ is a random fixed point of F. However the real valued function $y(\omega)$

defined as

$$y(\omega) = \begin{cases} 1, & \text{for} \quad \omega \in E, \\ -1, & \text{for } otherwise, \end{cases} \tag{1.6.9}$$

is a wide sense solution of the fixed point equation $F(\omega, x(\omega)) = x(\omega)$ without being random fixed point of F.

In this chapter, random fixed point of *Ćirić* type contractive mapping over a separable Banach space equipped with complete probability measure space has been investigated. The results obtained in this paper are stochastic generalization of important deterministic results in the literature. Finally, an application has been shown to ensure the existence of random solution of a non-linear stochastic integral equation of the Hammerstein type.

1.6.6 Chapter VII

In this chapter, our aim is to apply the theory of fixed points to some problems of applied interest. In particular, we would apply the mathematical methods of the theory of fixed points to some problems of physical interest e.g., problems of homogeneous and isotopic turbulence. We shall first mention here an important research article presented by H. K. Moffatt ([55]) e.g., "Topological Approach to problems of Vertex Dynamics and Turbulence" at the Fifth Beer-Sheva International seminar on 'Magnetohydrodynamics Flow and Turbulence', Ben-Gurion University of Negev, Beer-Sheva, Israel, March 2-6,1987.

Moffatt ([55]) introduced first the steady solutions of Euler equations in the context of turbulence due to their significance in representing the fixed points of the governing dynamical system evolving at high Reynolds number in the function space of solenoidal vector fields of finite energy density. These flows have a characteristic structures which may be interpreted in the language of turbulence. For example, Kelvin-Helmotz instability of the vortex sheets provides an inertial range spectrum and intermittent dissipative structures.

Moffatt ([55]) discussed four features of turbulence flow e.g.,

A. Coherent structures: there are identifiable structures of length scale and time scale large compared with the scale characterizing the energy-containing eddies of the turbulence.

B. Intermittency and dissipation: in this case, the local rate of dissipation of energy per unit mass $\epsilon =< \nu(\frac{\partial u_i}{\partial x_j})^2 >$ is intermittent. This means that as $\Re e \to \infty$, then $\epsilon(x,t)$ becomes increasingly spiky.

C. Inertial-range spectrum: The universal equilibrium theory of Kolmogorv is obeyed at the inertial subrange spectrum

$$E(k) = C\overline{\epsilon}^{\frac{2}{3}} k^{-\frac{5}{3}}$$

for $k_0 \ll k \ll \Re e^{\frac{3}{4}} k_0$, where k_0 is a wave number characteristic of the energy-containing eddies.

D. Enstrophy Production: The enstrophy $< w^2 >$, where $w(x,t)$ is the vorticity field, given by the relation

$$< w^2 >= \frac{\overline{\epsilon}}{\nu} \sim \Re e < u^2 > l_0^{-2}$$

for a given level of $\overline{\epsilon}$ of energy dissipation

$$< w^2 > \to \infty \quad as \quad \nu \to \infty.$$

The equation of enstophy is

$$\frac{d}{dt} < w^2 >=< w_i w_j \frac{\partial u_i}{\partial u_j} > -\nu < (\nabla \times w)^2 >$$

and production term $< w_i w_j \frac{\partial u_i}{\partial u_j} >$ is associated with stretching of vertex lines as these an distorted by the flow.

In particular, we would concentrate our attention to the roles of fixed points to the behavior of the solutions of the problems of isotropic homogeneous turbulence in fluid mechanics. Here in the present case, we would deal with the consequences of the applications of fixed points to the well known equation of $Von-K\acute{a}rm\acute{a}n-Howarth$([81]) equation of homogeneous isotopic turbulence.

Speziale and Bernard ([77]) pointed out that generally two types of asymptotic solutions are obtainable for isotopic turbulence e.g. the turbulent kinetic energy decay as $K \sim t^{-1}$ and $K \sim t^{-\alpha}$, where $\alpha > 1$ with explicit dependence on the initial conditions. It is to be mentioned that by a fixed point analysis and numerical integration of the exact one point equations, the turbulent kinetic energy $K \sim t^{-1}$ which is consistent with asymptotic validity of high Reynolds number solutions. Here we will show small departure from a state of complete self-preservation may be accepted and the nature of the fixed points.

1.7 Concluding remarks

This Ph.D. thesis entitled "FIXED POINTS FOR A CLASS OF OPERATORS OVER DIFFERENT TOPOLOGICAL STRUCTURED SPACES" is therefore a union of seven chapters (including Introductory Chapter) entirely dealing with problems of fixed points in different spatial structures each inviting some specific class of operators acting upon the same and also I have included in the last (Chapter- VII) on the application of theories of fixed point to a problem in applied interest. Individual chapters here stands self-complete and this warrants inclusion of reference list at the end of the chapters exclusively connected with the materials presented in the chapters only. So I sincerely hope that reading of the thesis would thereby be more attractive.

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Chapter 2

Approximate coincidence point for two non linear mappings over metric spaces

2.1 Introduction

Fixed point theory has been an important tool for solving various problems in non-linear functional analysis and as well as useful for proving the existence theorems for nonlinear differential and integral equations. However, in many practical situations, the conditions in the fixed point theorems are too strong and so the existence of a fixed point is not guaranteed. In that situation, one can consider nearly fixed points what we call as approximate fixed points. By an approximate fixed point x of a function f we mean in a sense that $f(x)$ is 'near to' x. The study of approximate fixed point theorems is equally interesting to that of fixed point theorems. Motivated by the article of Tijs, Torre and R. Brănzei([23]), Berinde([3]) established some fundamental approximate fixed point theorems in a metric space. Recently, Dey et al.([6]) investigated the approximate fixed point theorems for Reich operator which in turn generalize the results of Berinde([3]), showing that the existence of approximate fixed point for Reich operator is still obtainable in spite of the completeness of the underlying space being withdrawn.

On the otherhand, development in coincidence point theory is as busy as that of in metric fixed point theory. It was due to Sessa ([20]-[21]) who introduced the notion of weakly commutating maps in metric spaces in searching of their common fixed point and Jungck ([13]) supplemented the term of compatible mappings by generalizing the concept of weak commutativity. Subsequently Jungck and Rhoades ([14]) obtained a sufficient condition for a pair of self mappings which are weakly compatible on the assumption of their commutativity at their coincidence points. Assuming these concepts, many researchers had succeeded to get coincidence points for a different classes of mappings over partially ordered metric spaces, quasi metric spaces, cone metric spaces and in a topological vector spaces. As a survey of current literatures on coincidence point theory, one can refer to Du ([9]), Choudhury ([5]), Beg ([1]).

In this chapter, we study the existence of approximate coincidence point together with the property that the ϵ-fixed points are concentrated in a set with the diameter tends to zero if $\epsilon \to 0$. Also we discuss some results on approximate coincidence point of two nonlinear mappings used by Geraghty([11]) in 1973 and by Mizoguchi and Takahashi([16])in 1989. The mappings used by Geraghty ([11]) and by Mizoguchi and Takahashi ([16]) are called Geraghty function and $\mathcal{MT}$-function respectively.

2.2 Preliminaries

Definition 2.2.1. ([17]) Let (X, d) be a metric space and $f : X \to X$, let $\epsilon > 0$ be arbitrary. Take $x_0 \in X$. Then x_0 is an ϵ-fixed point (approximate fixed point) of f if $d(f(x_0), x_0) < \epsilon$.

The set of all ϵ-fixed points of f, for a given ϵ, is denoted by

$$F_\epsilon(f) = \{x \in X | d(f(x), x) < \epsilon\}.$$

Definition 2.2.2. ([17]) Let $f : X \to X$. Then f has the approximate fixed point property if

$$\forall \epsilon > 0, F_\epsilon(f) \neq \phi.$$

Now we define approximate coincidence point for two self-maps.

Definition 2.2.3. Let (X, d) be a metric space, $f, g : X \to X$ be two single-valued self-maps. The maps f and g are said to have approximate coincidence point property provided

$$\inf_{x \in X} d(fx, gx) = 0$$

or, equivalently, for any $\epsilon > 0$, there exists $z \in X$ such that

$$d(fz, gz) < \epsilon.$$

The set of all approximate coincidence point of f and g is denoted by $\mathcal{COP}_\epsilon(f, g)$.

2.3 Main Results

In this section, we have established existence of some results concerning approximate coincidence point for various types of nonlinear contractive maps in the setting of general metric spaces. For this purpose, we first define approximate coincidence point for two self-maps in a metric space and prove some results on approximate coincidence points by using the idea of Geraghty-type contractive condition ([11]).

In 1973, M. Geraghty ([11]) introduced the following class of functions as follows: Let $\mathcal{S}$ denote the class of real functions $\beta : [0, \infty) \to [0, 1)$ satisfying condition

$$\beta(t_n) \to 1 \text{ implies } t_n \to 0 \tag{2.3.1}$$

An example of a function in $\mathcal{S}$ may be given by $\beta(t) = e^{-2t}$ for $t > 0$ and $\beta(0) \in [0, 1)$. We now prove our result using this β-function.

Theorem 2.3.1. *Let (X, d) be a metric space and let $f, g : X \to X$ be two single valued self-mappings such that $f(X) \subset g(X)$ satisfying*

$$d(fx, fy) \leq \beta\left(d(gx, gy)\right) d(gx, gy) \tag{2.3.2}$$

for all $x, y \in X$ and $\beta \in \mathcal{S}$. Then the following statements hold:

(A_1) *f and g have the approximate coincidence point property on X*

i.e. $\inf_{x\in X} d(fx, gx) = 0$.

(A$_2$) *There exists a sequence* $\{z_n\}$ *in* (X, d) *such that* $\{z_n\}$ *is a Cauchy sequence and* $\lim_{n\to\infty} d(z_n, z_{n+1}) = \inf_{x\in X} d(fx, gx) = 0$.

Proof. Let $x_0 \in X$ be arbitrary. Since $f(X) \subset g(X)$, we can choose $x_1 \in X$ such that $gx_1 = fx_0$. Continuing this process, we obtain a sequence $\{x_n\}$ in X as follows:

$$gx_{n+1} = fx_n, \qquad n = 0, 1, 2, ... \tag{2.3.3}$$

We will suppose $d(fx_n, fx_{n+1}) > 0$ for all $n \in \mathbb{N}$, since if $fx_n = fx_{n+1}$ for some n, then $d(gx_{n+1}, fx_{n+1}) = d(fx_n, fx_{n+1}) = 0 < \epsilon$ implies that f, g have approximate coincidence point x_{n+1} and this completes the proof. So we suppose that $d(fx_n, fx_{n+1}) > 0$. Then by (2.3.1)

$$\begin{aligned} d(fx_{n+1}, fx_{n+2}) &\leq \beta\left(d(gx_{n+1}, gx_{n+2})\right) d(gx_{n+1}, gx_{n+2}) \\ &= \beta\left(d(fx_n, fx_{n+1})\right) d(fx_n, fx_{n+1}) \\ &< d(fx_n, fx_{n+1}) \end{aligned} \tag{2.3.4}$$

and so $d(fx_n, fx_{n+1}) < d(fx_{n-1}, fx_n)$. Hence $\{d(fx_n, fx_{n+1})\}$ is a strictly decreasing sequence which is bounded below and hence converging to some $q \geq 0$. Suppose that $q > 0$. Then using (2.3.4) we get

$$\frac{d(fx_{n+1}, fx_{n+2})}{d(fx_n, fx_{n+1})} \leq \beta\left(d(fx_n, x_{n+1})\right) < 1. \tag{2.3.5}$$

Now passing on limit as $n \to \infty$ on (2.3.5) we get

$$\lim_{n\to\infty} \beta\left(d(fx_n, x_{n+1})\right) \to 1. \tag{2.3.6}$$

Now using property of the function β, we conclude that $\lim_{n\to\infty} d(fx_n, x_{n+1}) = 0$. Assume $z_n = fx_n$ for $n \in \mathbb{N}$. So $\lim_{n\to\infty} d(z_n, z_{n+1}) = 0$. It implies that $\inf_{x\in X} d(fx, gx) \leq d(fx_n, gx_n) \leq d(z_{n-1}, z_n)$ for all $n \in \mathbb{N}$. Since $d(z_n, z_{n+1}) \to 0$ as $n \to \infty$, it follows from above that $\inf_{x\in X} d(fx, gx) = 0$. So f and g have approximate coincidence point x in X. So (A_1) follows.

Now it suffices to prove that $\{z_n\}$ is a Cauchy sequence in (X, d). Let $\lambda = \beta(d(z_n, z_{n+1}))$. Then using the property of β and from (2.3.5), we have $0 \leq \lambda < 1$. Again using (2.3.5) we get $d(z_{n+1}, z_{n+2}) \leq \lambda d(z_n, z_{n+1})$. In this process, we obtain

$$d(z_{n+1}, z_{n+2}) \leq \lambda^n d(z_1, z_2) \tag{2.3.7}$$

For $m, n \in \mathbb{N}$ with $m > n$, it follows from (2.3.7) that

$$d(z_n, z_m) \leq \sum_{i=n}^{m-1} d(z_i, z_{i+1}) \leq \frac{\lambda^{n-1}}{1-\lambda} d(z_1, z_2).$$

Since $0 \leq \lambda < 1$, $d(z_n, z_m) \to 0$ as $n \to \infty$ and so $\{z_n\}$ is a Cauchy sequence. Hence (A_2) follows. □

Example 2.3.2. Let $X = [1, \infty)$ with d be usual metric and $f, g : [1, \infty) \to [1, \infty)$ be defined by

$$f(x) = \frac{1}{4x}, g(x) = \frac{1}{2x}$$

for all $x \in X$.

Also take $\beta(t) = e^{-2t}$ for $t > 0$ and $\beta(0) \in [0, 1)$. Then one can check the inequality (2.3.2) is satisfied with $x > y$, $x, y \in X$. Again it is easy to check that all the conditions of Theorem (2.3.1) are satisfied having the approximate coincidence point property. In fact $d(fx_0, gx_0) = \left|\frac{1}{4x_0} - \frac{1}{2x_0}\right| = \frac{1}{4x_0} < \epsilon$; if we select $x_0 \in [1, \infty)$ such that $x_0 > \frac{1}{4\epsilon}$ with $0 < \epsilon < \frac{1}{4}$. On the contrary, it is clear that f and g have no coincidence point in $[1, \infty)$.

In 1989, Mizoguchi and Takahashi ([16]) introduced $\mathcal{MT}$-function as following:

A function $\varphi : [0, \infty) \to [0, 1)$ is said to be an $\mathcal{MT}$-function, if

$$\limsup_{s \to t^+} \varphi(s) < 1 \quad \text{for all } t \in [0, \infty).$$

It is obvious that if $\varphi : [0, \infty) \to [0, 1)$ is a non-decreasing function or a non-increasing function, then φ is an $\mathcal{MT}$-function.

Definition 2.3.3. Let (X, d) be a metric space and $\varphi : [0, \infty) \to [0, 1)$ is a $\mathcal{MT}$-function. Then $T : X \to X$ is said to be a $\mathcal{MT}$-type mapping if

$$d(Tx, Ty) \leq \varphi(d(x, y)) d(x, y).$$

Using this $\mathcal{MT}$ function Mizoguchi and Takahashi ([16]) proved a fixed point theorem for multivalued mapping, which is a generalization of Nadler's fixed point theorem which extends the Banach contraction mapping to multivalued mappings, but its primitive proof is different and was given by Granas and Dugundji

([12]). But we only restrict our discussion here for single valued mapping. In this aspect, we formulate our next result using $\mathcal{MT}$-function. For the properties and characterizations of $\mathcal{MT}$-function one can see (Wei-Shih Du([8]) and Du et al.([9])) for details.

Now we establish the following approximate coincidence point property using the concept of Mizoguchi and Takahashi ($\mathcal{MT}$)-type mappings.

Theorem 2.3.4. *Let (X, d) be a metric space, $f, g : X \to X$ be two single valued self-maps such that $f(X) \subset g(X)$ satisfying:*

$$d(fx, fy) \leq \varphi\left(d(gx, gy)\right) d(gx, gy) \tag{2.3.8}$$

for all $x, y \in X$ where $\varphi : [0, \infty) \to [0, 1)$ is a $\mathcal{MT}$-function.
Then the following statements hold:
(A_1') *f and g have the approximate coincidence point property on X i.e.*
$\inf_{x \in X} d(fx, gx) = 0.$
(A_2') *There exists a sequence $\{z_n\}$ in (X, d) such that $\{z_n\}$ is a Cauchy sequence and $\lim_{n\to\infty} d(z_n, z_{n+1}) = \inf_{x\in X} d(fx, gx) = 0.$*

Proof. Proof is similar to that of Theorem(2.3.1) and left out. The only difference lies in the character of the β-function and $\mathcal{MT}$-function which are supplementary to each other. □

Example 2.3.5. Let $X = [0, \infty)$ with d be usual metric and $f, g : [1, \infty) \to [1, \infty)$ be defined by

$$f(x) = \frac{1}{4x}, g(x) = \frac{1}{2x}$$

for all $x \in X$.
If we take $\varphi : [0, \infty) \to [0, 1)$ defined by $\varphi(x) = \frac{2}{3}$, then all the conditions of the Theorem (2.3.4) are satisfied.

It is straight forward to arrive at the following corollaries:

Corollary 2.3.6. *Let (X, d) be a metric space, $f : X \to X$ be a single valued self-map satisfying:*

$$d(fx, fy) \leq \varphi\left(d(x, y)\right) d(x, y) \tag{2.3.9}$$

for all $x, y \in X$ *where* $\varphi : [0, \infty) \to [0, 1)$ *is a* $\mathcal{MT}$*-function.*
Then the following statements hold:
(B_1) f *has the approximate fixed point on* X *i.e.* $\inf_{x \in X} d(x, fx) = 0$.
(B_2) *There exists a sequence* $\{z_n\}$ *in* (X, d) *such that* $\{z_n\}$ *is a Cauchy sequence and* $\lim_{n\to\infty} d(z_n, z_{n+1}) = \inf_{x \in X} d(x, fx) = 0$.

Corollary 2.3.7. *Let* (X, d) *be a metric space and let* $f : X \to X$ *be two single valued self-mapping satisfying*

$$d(fx, fy) \leq \beta\left(d(x, y)\right) d(x, y) \tag{2.3.10}$$

for all $x, y \in X$ *and* $\beta \in \mathcal{S}$. *Then the following statements hold:*
(B_1') f *has the approximate fixed point on* X *i.e.* $(\inf_{x \in X} d(fx, gx) = 0)$.
(B_2') *There exists a sequence* $\{z_n\}$ *in* (X, d) *such that* $\{z_n\}$ *is a Cauchy sequence and* $\lim_{n\to\infty} d(z_n, z_{n+1}) = \inf_{x \in X} d(x, fx) = 0$.

Remark 2.3.8. Corollary (2.3.6) and Corollary (2.3.7) are the generalizations of Banach Contraction Principle in approximate version.

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[0]Some of the materials presented in this chapter appeared in our paper (see-[6]) entitled "Approximate coincidence point of two nonlinear mappings", Journal of Mathematics, Volume 2013, Article ID 962058, 4 pages.

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Chapter 3

Fixed point of $\alpha - \psi$ multivalued mapping on a cone metric space

3.1 Introduction

Several attempts have been carried out by many researchers to generalize the metric spaces by considering vector valued cone metrics whose values are in an ordered real Banach space until Huang Long-Guang and Zhang Xian([10]) had been able to introduce the concept of cone metric space in 2007. They also proved some analogues of basic fixed point results like Banach contraction principle and Kannan fixed point theorem over the such spaces. Following the literatures in this area many fixed point theorems were obtained by Abbas ([1]), Ilić ([12]), Rezapour ([27]), Turkoglu([31]), Vetro ([32]) and by some others over cone metric spaces. Also topological questions in cone metric spaces were studied by Turkoglu and Abhuloha ([31]), where it was proved that every cone metric space is first countable by showing the equivalent properties of continuity vis a vis sequential continuity and compactness vis a vis sequential compactness. Motivated by these works, researchers like Karapinar ([17]-[20]), Khamsi([22]), Rezapour([26]), Du([8]) etc. analyzed the existence of fixed point for different types of contractive mappings. Also results relating to common fixed point theorem for a class of mapping have also been generalized by Abbas ([1]), Vetro ([32]), Ilić ([12]) and by some others. In this chapter, we have established some fixed point theorems for $\alpha - \psi$- set valued contractive mappings by using the concept of $\alpha - \psi$- Geraghty (see- Karapinar([17])) contraction over cone metric space which in turn generalize and unify the earlier results.

Definition 3.1.1. Let E be real Banach space and P, a subset of E. Then P is said to be a cone whenever

(i) P is closed, non-empty and $P \neq \{0\}$,

(ii) $ax + by \in P$ for all $x, y \in P$ and for non-negative real numbers a, b ,

(iii) $P \cap (-P) = \{0\}$.

For a given cone $P \subseteq E$, we can define the partial ordering $' \leq'$ with respect to P by $a \leq b$ if and only if $b - a \in P$, $a < b$ stands for $a \leq b$ but $a \neq b$ and while $a \ll b$ stands for $b - a \in IntP$, where $IntP$ denotes the interior of P.
The cone P is said to be a normal cone if there exists a real number $k > 0$ such that $\forall x, y \in E$, $0 \leq x \leq y \Rightarrow \|x\| \leq k\|y\|$. The least positive number k satisfying the above is called the normal constant of P. Clearly $k \geq 1$.
In the following we always suppose that E is a normed space and P is a cone in E with normal constant $k = 1$, $IntP \neq \phi$ and " $\leq$ " is partial ordering with respect to P.

3.2 Preliminaries

Definition 3.2.1. [26] Let X be a non-empty set. Suppose the mapping $d : X \times X \rightarrow E$ satisfies

(i) $0 \leq d(x, y), \forall x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$,

(ii) $d(x, y) = d(y, x)$, $\forall x, y \in X$,

(iii) $d(x, y) \leq d(x, z) + d(z, y)$, $\forall x, y, z \in X$.

Then we say that d is a cone metric on X and (X, d) is a cone metric space.

Example 3.2.2. [26] Let $E = \mathbb{R}^2$, $P = \{(x, y) \in E : x, y \geq 0\}$, $X = \mathbb{R}$ and $d : X \times X \to E$ defined by $d(x, y) = (|x-y|, \alpha|x-y|)$, where $\alpha \geq 0$ is a constant. Then (X, d) is a cone metric space with the normal constant of P is $k = 1$.

Definition 3.2.3. [26] Let (X, d) be a cone metric space, $x \in X$ and $\{x_n\}$ be a sequence in X. Then

(A) $\{x_n\}$ converges to $x \in X$ whenever for every $c \in E$ with $0 \ll c$ there is a natural number N such that $d(x_n, x) \ll c$ for all $n \geq N$. We denote it by $\lim_{n\to\infty} x_n = x$ or $x_n \to x (n \to \infty)$.

(B) $\{x_n\}$ is a Cauchy sequence whenever for every $c \in E$ with $0 \ll c$ there is a natural number N such that $d(x_n, x_m) \ll c$ for all $n, m \geq N$ or $d(x_n, x_m) \to 0 (n \to \infty)$.

(C) (X, d) is a complete cone metric space if every Cauchy sequence in X is convergent in X.

Definition 3.2.4. [26] Let (X, d) be a cone metric space and $B \subset X$,

(1) $b \in B$ is called an interior point of B whenever there is $0 \ll p$ such that $N(b, p) \subset B$, where

$$N(b, p) = \{y \in X : d(y, b) \ll p\}.$$

(2) A subset $A \subset X$ is called open if each element of A is an interior point of A.

The family $\mathfrak{B} = \{N(x, e) : x \in X, 0 \ll e\}$ is a sub-basis for a topology, called cone topology on X. We denote it by τ_c and the topology τ_c is Hausdorff and first countable.

We are mainly familiar with normal cones having normal constant equal to 1. But for each positive integer $k(> 1)$, there are cones with normal constant $k > 1$. Existence of non normal cones could also be found in Rezapour([27]). Before introducing our main results we first state the following lemmas.

Lemma 3.2.5. *[26] Let (X, d) be a cone metric space, P be a normal cone with normal constant equal to 1, and A be a compact set in (X, τ_c). Then for every $x \in X$, $\exists a_0 \in A$ such that $\|d(x, a_0)\| = \inf_{a\in A} \|d(x, a)\|$.*

Lemma 3.2.6. *[26] Let (X, d) be a cone metric space, P be a normal cone with normal constant equal to 1, and A, B are two compact sets in (X, τ_c). Then, $\sup_{x\in B} d^{/}(x, A) < \infty$, where, $d^{/}(x, A) = \inf_{a\in A} \|d(x, a)\|$.*

Definition 3.2.7. [26] Let (X, d) be a cone metric space, P be a normal cone with normal constant equal to 1, $\mathcal{H}_c(X)$ be the set of all compact subsets of (X, τ_c) and $A \in \mathcal{H}_c(X)$. then we define

$$h_A : \mathcal{H}_c(X) \to [0, \infty) \text{ by } h_A(B) = \sup_{x\in A} d^{/}(x, B). \tag{3.2.1}$$

and

$$d_H : \mathcal{H}_c(X) \times \mathcal{H}_c(X) \to [0, \infty) \text{ by } d_H(A, B) = \max\{h_A(B), h_B(A)\}. \tag{3.2.2}$$

Remark 3.2.8. [26] Let (X, d) be a cone metric space with normal constant equal to 1. Define $\rho : X \times X \to [0, \infty)$ by $\rho(x, y) = \|d(x, y)\|$. Then (X, ρ) is a metric space.

Remark 3.2.9. [26] For each $A, B \in \mathcal{H}_c(X)$ and $x, y \in X$, we have the following relations

(a) $d^{/}(x, A) \leq \|d(x, y)\| + d^{/}(y, A)$.

(b) $d^{/}(x, A) \leq d^{/}(x, B) + h_B(A)$.

(c) $d^{/}(x, A) \leq \|d(x, y)\| + d^{/}(y, B) + h_B(A)$.

Definition 3.2.10. [7] Let $T : X \to X$ be a map and $\alpha : X \times X \to \mathbb{R}$ be a function. Then T is said to be α-admissible if $\alpha(x, y) \geq 1$ implies $\alpha(Tx, Ty) \geq 1$, for all $x, y \in X$.

Lemma 3.2.11. *[7] An α-admissible map T is triangular α-admissible if $\alpha(x, y) \geq 1$ and $\alpha(y, z) \geq 1$ implies $\alpha(x, z) \geq 1$, for all $x, y, z \in X$.*

Lemma 3.2.12. *[17] Let $T : X \to X$ be a triangular α-admissible map. Assume that there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq 1$. Define a sequence $\{x_n\} \in X$ by $x_{n+1} = Tx_n$. Then we have $\alpha(x_n, x_m) \geq 1$ for all $m, n \in \mathbb{N}$ with $n < m$.*

Definition 3.2.13. [15] Let Ψ be the class of all functions ψ where $\psi : [0, \infty) \to [0, \infty)$ satisfies the following conditions:

(i) ψ is non-decreasing.

(ii) ψ is sub-additive i.e. $\psi(s+t) \leq \psi(s) + \psi(t)$ for all $s, t \in [0, \infty)$.

(iii) ψ is continuous.

(iv) $\psi(t) = 0$ if and only if $t = 0$.

3.3 Main Results

Definition 3.3.1. Let $T : X \to \mathcal{H}_c(X)$ be a map and $\alpha : \mathcal{H}_c(X) \times \mathcal{H}_c(X) \to \mathbb{R}$ be a function. Then T is said to be Hausdorff α-admissible if $\alpha(A, B) \geq 1$ implies $\alpha(TA, TB) \geq 1$, for all $A, B \in \mathcal{H}_c(X)$.

Definition 3.3.2. A Hausdorff α-admissible map T is said to be triangular α-orbital contraction if $\alpha(A, B) \geq 1$ and $\alpha(B, C) \geq 1$ implies $\alpha(A, C) \geq 1$, for all $A, B, C \in \mathcal{H}_c(X)$

Lemma 3.3.3. *Let $T : X \to \mathcal{H}_c(X)$ be a triangular α-orbital contraction map. Assume that there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq 1$. Define a sequence $\{x_n\}$ by $x_{n+1} \in Tx_n$. Then we have $\alpha(x_n, x_m) \geq 1$ for all $m, n \in \mathbb{N}$ with $n < m$.*

Proof. The proof is straight froward. □

We denote

$$(d_H)_\psi(A, B) = \psi(d_H(A, B)) \qquad \textit{and} \qquad d^{/}_\psi(x, y) = \psi(d^{/}(x, y)).$$

Definition 3.3.4. Let (X, d) be a cone metric space and let $\alpha : \mathcal{H}_c(X) \times \mathcal{H}_c(X) \to \mathbb{R}$ be a function. A map $T : X \to \mathcal{H}_c(X)$ is called $\alpha - \psi$-Banach type contraction if

$$\alpha(x, y)\psi(d_H(Tx, Ty)) \leq c\psi(d^{/}(x, y)) \tag{3.3.1}$$

where $\psi \in \Psi$ and $x, y \in X$ and $0 < c < 1$.

Theorem 3.3.5. *Let (X, d) be a complete cone metric space with normal constant equal to 1. Let $\alpha : \mathcal{H}_c(X) \times \mathcal{H}_c(X) \to \mathbb{R}$ be a function and the multifunction $T : X \to \mathcal{H}_c(X)$ satisfies the following conditions:*

(1) $\alpha(x,y)\psi(d_H(Tx,Ty)) \leq c\psi(d^{/}(x,y))$ *where* $\psi \in \Psi$ *and* $x,y \in X$ *and* $0 < c < 1$.

(2) T *is triangular* α*-orbital contraction.*

(3) there exists $x_1 \in X$ *such that* $\alpha(x_1, Tx_1) \geq 1$.

Then T *has a fixed point in* X.

Proof. As $x_1 \in X$ we have $\alpha(x_1, Tx_1) \geq 1$. Then we can define a sequence $\{x_n\} \in X$ by $x_{n+1} \in Tx_n$ for all $n \in \mathbb{N}$. If $x_{n_0} = x_{n_0+1}$ for some $n_0 \in \mathbb{N}$, then x_{n_0} is a fixed point. So the proof is done in this case. So let $x_n \neq x_{n+1}$, $\forall n \in \mathbb{N}$. Then by Lemma(3.3.3), we get $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$. As $x_1 \in X$, again by Lemma(3.2.6), $\exists x_2 \in Tx_1$ such that $d^{/}(x_1, Tx_1) = \|d(x_1, x_2)\|$. In this way if $x_n \in Tx_{n-1}$, then $\exists x_{n+1} \in Tx_n$ such that $d^{/}(x_n, Tx_n) = \|d(x_n, x_{n+1})\|$. Now

$$\begin{aligned} \psi(\|d(x_n, x_{n+1})\|) &= \psi(d^{/}(x_n, Tx_n)) \leq \psi(h_{Tx_{n-1}}(Tx_n)) \leq \psi(d_H(Tx_{n-1}, Tx_n)) \\ &\leq \alpha(x_{n-1}, x_n)\psi(d_H(Tx_{n-1}, Tx_n)) \leq c(\psi(d^{/}(x_{n-1}, x_n))) \\ &\leq c(\psi(\|d(x_{n-1}, x_n)\|)) \qquad (3.3.2) \end{aligned}$$

As $c < 1$,

$$\psi(\|d(x_n, x_{n+1})\|) < (\psi(\|d(x_{n-1}, x_n)\|)). \qquad (3.3.3)$$

As ψ is non-decreasing we get,

$$\|d(x_n, x_{n+1})\| < \|d(x_{n-1}, x_n)\| \qquad (3.3.4)$$

Then $\{\|d(x_n, x_{n+1})\|\}$ is decreasing sequence of real numbers which is also bounded below. Hence the sequence is convergent and let it converges to $r \geq 0$. We claim that $r = 0$. Otherwise if $r > 0$, we have $\|d(x_n, x_{n+1})\| \to r$. As ψ is continuous then $\psi(\|d(x_n, x_{n+1})\|) \to \psi(r)$, as $n \to \infty$.
So from (3.3.2) we say that $\psi(r) < c\psi(r)$ i.e. $c > 1$, a contradiction. So, $r = 0$. Hence the sequence $\{\|d(x_n, x_{n+1})\|\}$ converges to 0.
But,

$$\begin{aligned} &\psi(\|d(x_n, x_{n+1})\|) < c(\psi(\|d(x_{n-1}, x_n)\|)) \\ &\leq ... \leq c^{n-1}(\psi(\|d(x_1, x_2)\|)) \qquad (3.3.5) \end{aligned}$$

Hence,

$$\begin{aligned}
\psi(\|d(x_n, x_m)\|) \quad & \leq \psi(\sum_{i=m}^{n-1} \|d(x_i, x_{i+1})\|) \\
& \leq \sum_{i=m}^{n-1} \psi(\|d(x_i, x_{i+1})\|) \\
& \leq (c^{n-2} + ... + c^{m-1})\psi(\|d(x_1, x_2)\|) \\
& \leq \frac{c^{m-1}}{1-c}\psi(\|d(x_1, x_2)\|) \qquad (3.3.6)
\end{aligned}$$

So

$$\lim_{m,n\to\infty} \psi(\|d(x_n, x_m)\|) = 0 \qquad (3.3.7)$$

Consequently,

$$\lim_{m,n\to\infty} \|d(x_n, x_m)\| = 0 \qquad (3.3.8)$$

So $\{x_n\}$ is a Cauchy sequence in X. As X is complete, $\lim_{n\to\infty} x_n = \xi \in X$. As $\alpha(x_n, x_m) \geq 1$ for $n < m$, we see that $\alpha(x_n, \xi) \geq 1$ for all n.

$$d^{/}(\xi, T\xi) \leq d^{/}(\xi, Tx_n) + h_{Tx_n}(T\xi) \leq d^{/}(\xi, Tx_n) + d_H(Tx_n, T\xi) \qquad (3.3.9)$$

So,

$$\begin{aligned}
\psi(d^{/}(\xi, T\xi)) \quad & \leq \psi(d^{/}(\xi, Tx_n)) + \psi(d_H(Tx_n, T\xi)) \\
& \leq \psi(d^{/}(\xi, Tx_n)) + \alpha(x_n, \xi)\psi(d_H(Tx_n, T\xi)) \\
& \leq \psi(d^{/}(\xi, Tx_n)) + c(\psi(d^{/}(x_n, \xi)))
\end{aligned} \qquad (3.3.10)$$

As $n \to \infty$,

$$\psi(d^{/}(\xi, T\xi)) = 0 \qquad \text{impaling that } d^{/}(\xi, T\xi) = 0 \qquad (3.3.11)$$

and hence $\xi \in T\xi$, showing that ξ is a fixed point of T. □

Definition 3.3.6. Let (X, d) be a cone metric space and let $\alpha : \mathcal{H}_c(X) \times \mathcal{H}_c(X) \to \mathbb{R}$ be a function. A map $T : X \to \mathcal{H}_c(X)$ is called $\alpha - \psi$-Kannan type contraction if

$$\alpha(x, y)\psi(d_H(Tx, Ty)) \leq c\psi(d^{/}(Tx, x) + d^{/}(y, Ty)) \qquad (3.3.12)$$

where $\psi \in \Psi$ and $x, y \in X$ and $0 < c < \frac{1}{2}$.

Theorem 3.3.7. *Let (X,d) be a complete cone metric space with normal constant equal to 1. Let $\alpha : \mathcal{H}_c(X) \times \mathcal{H}_c(X) \to \mathbb{R}$ be a function and the multifunction $T : X \to \mathcal{H}_c(X)$ satisfies the following conditions:*

(1) $\alpha(x,y)\psi(d_H(Tx,Ty)) \leq c\psi(d^{/}(x,Tx) + d^{/}(y,Ty))$ where $\psi \in \Psi$ and $x,y \in X$ and $0 < c < \frac{1}{2}$.

(2) T is triangular α-orbital contraction.

(3) there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq 1$.

Then T has a fixed point in X.

Proof. For $x_1 \in X$ we have $\alpha(x_1, Tx_1) \geq 1$. Then we can define a sequence $\{x_n\} \in X$ by $x_{n+1} \in Tx_n$ for all $n \in \mathbb{N}$. If $x_{n_0} = x_{n_0+1}$ for some $n_0 \in \mathbb{N}$. Then x_{n_0} is a fixed point and the proof is done. So let $x_n \neq x_{n+1}$, $\forall n \in \mathbb{N}$, then by Lemma(3.3.3), we have $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$. As $x_1 \in X$, again by Lemma(3.2.6), $\exists x_2 \in Tx_1$ such that $d^{/}(x_1, Tx_1) = \|d(x_1,x_2)\|$. In this way if $x_n \in Tx_{n-1}$, $\exists x_{n+1} \in Tx_n$ such that $d^{/}(x_n, Tx_n) = \|d(x_n, x_{n+1})\|$.

$$\begin{aligned}
\psi(\|d(x_n,x_{n+1})\|) &= \psi(d^{/}(x_n,Tx_n)) \leq \psi(h_{Tx_{n-1}}(Tx_n)) \leq \psi(d_H(Tx_{n-1},Tx_n)) \\
&\leq \alpha(x_{n-1},x_n)\psi(d_H(Tx_{n-1},Tx_n)) \\
&\leq c\psi(d^{/}(Tx_{n-1},x_{n-1}) + d^{/}(Tx_n,x_n)) \\
&\leq c(\psi(\|d(x_{n-1},x_n)\|) + \psi(\|d(x_{n+1},x_n)\|)) \qquad (3.3.13)
\end{aligned}$$

$$\psi(\|d(x_n,x_{n+1})\|) \leq (\frac{c}{1-c})\psi(\|d(x_{n-1},x_n)\|) \leq p\psi(\|d(x_{n-1},x_n)\|) \qquad (3.3.14)$$

where $p = (\frac{c}{1-c}) < 1$. So we can say,

$$\psi(\|d(x_n,x_{n+1})\|) \leq \psi(\|d(x_{n-1},x_n)\|) \qquad (3.3.15)$$

As ψ is non-decreasing,

$$\|d(x_n,x_{n+1})\| < \|d(x_{n-1},x_n)\| \qquad (3.3.16)$$

Thus $\{\|d(x_n,x_{n+1})\|\}$ is a decreasing sequence and also bounded below. Hence the sequence is convergent and by similar argument as in proof of Theorem (3.3.5),

the sequence $\{\|d(x_n, x_{n+1})\|\}$ converges to 0, as $n \to \infty$.
But by routine calculation

$$\begin{aligned} &\psi(\|d(x_n, x_{n+1})\|) < p(\psi(\|d(x_{n-1}, x_n)\|)) \\ &\leq ... \leq p^{n-1}(\psi(\|d(x_1, x_2)\|)) \end{aligned} \tag{3.3.17}$$

So,

$$\begin{aligned} \psi(\|d(x_n, x_m)\|) \quad &\leq \psi(\sum_{i=m}^{n-1} \|d(x_i, x_{i+1})\|) \\ &\leq \sum_{i=m}^{n-1} \psi(\|d(x_i, x_{i+1})\|) \\ &\leq (p^{n-2} + ... + p^{m-1})\psi(\|d(x_1, x_2)\|) \\ &\leq \frac{p^{m-1}}{1-p}\psi(\|d(x_1, x_2)\|) \end{aligned} \tag{3.3.18}$$

So

$$\lim_{m,n\to\infty} \psi(\|d(x_n, x_m)\|) = 0 \tag{3.3.19}$$

which implies that,

$$\lim_{m,n\to\infty} \|d(x_n, x_m)\| = 0 \tag{3.3.20}$$

So $\{x_n\}$ is a Cauchy sequence in X. As X is complete we have $\lim_{n\to\infty} x_n = \xi \in X$. and consequently we get $\alpha(x_n, \xi) \geq 1$ for all n.

$$d^{/}(\xi, T\xi) \leq d^{/}(\xi, Tx_n) + h_{Tx_n}(T\xi) \leq d^{/}(\xi, Tx_n) + d_H(Tx_n, T\xi) \tag{3.3.21}$$

So,

$$\begin{aligned} \psi(d^{/}(\xi, T\xi)) \quad &\leq \psi(d^{/}(\xi, Tx_n)) + \psi(d_H(Tx_n, T\xi)) \\ &\leq \psi(d^{/}(\xi, Tx_n)) + \alpha(x_n, \xi)\psi(d_H(Tx_n, T\xi)) \\ &\leq \psi(d^{/}(\xi, Tx_n)) + c(\psi(d^{/}(Tx_n, x_n) + d^{/}(T\xi, \xi))) \end{aligned} \tag{3.3.22}$$

$$\psi(d^{/}(\xi, T\xi)) \leq (\frac{1}{1-c})\psi(d^{/}(\xi, Tx_n)) + (\frac{c}{1-c})\psi(d^{/}(Tx_n, x_n)) \tag{3.3.23}$$

As $n \to \infty$, we get

$$\psi(d^{/}(\xi, T\xi)) = 0 \qquad \text{which implies } d^{/}(\xi, T\xi) = 0 \tag{3.3.24}$$

and hence $\xi \in T\xi$ i.e. ξ is a fixed point of T. □

Definition 3.3.8. Let (X, d) be a cone metric space and let $\alpha : \mathcal{H}_c(X) \times \mathcal{H}_c(X) \to \mathbb{R}$ be a function. A map $T : X \to \mathcal{H}_c(X)$ is called $\alpha - \psi$-Chatterjea type contraction if

$$\alpha(x, y)\psi(d_H(Tx, Ty)) \leq c\psi(d^{/}(Tx, y) + d^{/}(Ty, x)) \tag{3.3.25}$$

where $\psi \in \Psi$ and $x, y \in X$ and $0 < c < \frac{1}{2}$.

Theorem 3.3.9. *Let (X, d) be a complete cone metric space with normal constant equal to 1. Let $\alpha : \mathcal{H}_c(X) \times \mathcal{H}_c(X) \to \mathbb{R}$ be a function and the multifunction $T : X \to \mathcal{H}_c(X)$ satisfies the following conditions:*

(1) $\alpha(x, y)\psi(d_H(Tx, Ty)) \leq c\psi(d^{/}(Tx, y) + d^{/}(Ty, x))$ where $\psi \in \Psi$ and $x, y \in X$ and $0 < c < \frac{1}{2}$.

(2) T is triangular α-orbital contraction.

(3) there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq 1$.

Then T has a fixed point in X.

Proof. For $x_1 \in X$ we have $\alpha(x_1, Tx_1) \geq 1$. Then we can define a sequence $\{x_n\} \in X$ by $x_{n+1} \in Tx_n$ for all $n \in \mathbb{N}$. If $x_{n_0} = x_{n_0+1}$ for some $n_0 \in \mathbb{N}$, then x_{n_0} is a fixed point and hence the proof is complete. So let $x_n \neq x_{n+1}$, $\forall n \in \mathbb{N}$. Then by Lemma(3.3.3), we get that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$. As $x_1 \in X$ again by Lemma(3.2.6), we can find $x_2 \in Tx_1$ such that $d^{/}(x_1, Tx_1) = \|d(x_1, x_2)\|$. In this way if $x_n \in Tx_{n-1}$, $\exists x_{n+1} \in Tx_n$ such that $d^{/}(x_n, Tx_n) = \|d(x_n, x_{n+1})\|$.

$$\begin{aligned}
\psi(\|d(x_n, x_{n+1})\|) &= \psi(d^{/}(x_n, Tx_n)) \leq \psi(h_{Tx_{n-1}}(Tx_n)) \leq \psi(d_H(Tx_{n-1}, Tx_n)) \\
&\leq \alpha(x_{n-1}, x_n)\psi(d_H(Tx_{n-1}, Tx_n)) \\
&\leq c\psi(d^{/}(Tx_{n-1}, x_n) + d^{/}(Tx_n, x_{n-1})) \\
&\leq c(\psi(\|d(x_{n+1}, x_{n-1})\|)) \\
&\leq c(\psi(\|d(x_{n+1}, x_n)\| + \|d(x_n, x_{n-1})\|)) \\
&\leq c(\psi(\|d(x_{n+1}, x_n)\|) + \psi(\|d(x_n, x_{n-1})\|))
\end{aligned} \tag{3.3.26}$$

$$\psi(\|d(x_n, x_{n+1})\|) \leq (\frac{c}{1-c})\psi(\|d(x_{n-1}, x_n)\|) \leq p\psi(\|d(x_{n-1}, x_n)\|) \quad (3.3.27)$$

where $p = (\frac{c}{1-c}) < 1$. So we can say,

$$\psi(\|d(x_n, x_{n+1})\|) \leq \psi(\|d(x_{n-1}, x_n)\|) \quad (3.3.28)$$

As ψ is non-decreasing,

$$\|d(x_n, x_{n+1})\| < \|d(x_{n-1}, x_n)\| \quad (3.3.29)$$

In a similar fashion we can see that the sequence $\{\|d(x_n, x_{n+1})\|\}$ converges to 0. It is easy to see that,

$$\begin{aligned} &\psi(\|d(x_n, x_{n+1})\|) < p(\psi(\|d(x_{n-1}, x_n)\|)) \\ &\leq ... \leq p^{n-1}(\psi(\|d(x_1, x_2)\|)) \end{aligned} \quad (3.3.30)$$

So,

$$\begin{aligned} \psi(\|d(x_n, x_m)\|) \quad &< \psi(\sum_{i=m}^{n-1} \|d(x_i, x_{i+1})\|) \\ &\leq \sum_{i=m}^{n-1} \psi(\|d(x_i, x_{i+1})\|) \\ &\leq (p^{n-2} + ... + p^{m-1})\psi(\|d(x_1, x_2)\|) \\ &\leq \frac{p^{m-1}}{1-p}\psi(\|d(x_1, x_2)\|) \end{aligned} \quad (3.3.31)$$

Hence,

$$\lim_{m,n\to\infty} \psi(\|d(x_n, x_m)\|) = 0 \quad (3.3.32)$$

So,

$$\lim_{m,n\to\infty} \|d(x_n, x_m)\| = 0 \quad (3.3.33)$$

So $\{x_n\}$ is a Cauchy sequence in X. As X is complete we get that $\lim_{n\to\infty} x_n = \xi \in X$. and hence $\alpha(x_n, \xi) \geq 1$ for all n.

$$d^{/}(\xi, T\xi) \leq d^{/}(\xi, Tx_n) + h_{Tx_n}(T\xi) \leq d^{/}(\xi, Tx_n) + d_H(Tx_n, T\xi) \quad (3.3.34)$$

So,

$$\begin{aligned}\psi(d^{/}(\xi,T\xi)) \quad &\leq \psi(d^{/}(\xi,Tx_n))+\psi(d_H(Tx_n,T\xi))\\ &\psi(d^{/}(\xi,Tx_n))+\alpha(x_n,\xi)\psi(d_H(Tx_n,T\xi))\\ &\psi(d^{/}(\xi,Tx_n))+c(\psi(d^{/}(Tx_n,\xi)+d^{/}(T\xi,x_n)))\end{aligned} \tag{3.3.35}$$

$$\psi(d^{/}(\xi,T\xi))\leq(\frac{1+c}{1-c})\psi(d^{/}(\xi,Tx_n))+(\frac{c}{1-c})\psi(d^{/}(x_n,\xi)) \tag{3.3.36}$$

As $n\to\infty$, we get

$$\psi(d^{/}(\xi,T\xi))=0,\quad \text{which implies } d^{/}(\xi,T\xi)=0 \tag{3.3.37}$$

and consequently $\xi\in T\xi$ impaling that ξ is a fixed point of T. □

Following illustrative examples are given to support Theorem (3.3.5), Theorem (3.3.7) and Theorem (3.3.9) respectively which may be taken as some representative theorems derived by us.

Example 3.3.10. Let $X=[0,1]$, $E=\mathbb{R}^2$ and $P=\{(x,y)\in E: x,y\geq 0\}$. Define $d: X\times X\to E$ by $d(x,y)=(\frac{1}{2}|x-y|,\frac{1}{2}|x-y|)$, $x,y\in X$. Then (X,d) is a complete cone metric space with normal constant of P equal to 1. Define $T: X\to\mathcal{H}_c(X)$ by

$$T(x)=\begin{cases}\{0\} & \text{if } x\in[0,\frac{1}{2}];\\ [0,\frac{1}{2}(x-\frac{1}{2})^2] & \text{if } x\in(\frac{1}{2},1].\end{cases} \tag{3.3.38}$$

Define $\psi:[0,\infty)\to[0,\infty)$ by $\psi(t)=t$ for all $t\in[0,\infty)$. Let $\alpha:\mathcal{H}_c(X)\times\mathcal{H}_c(X)\to\mathbb{R}$ be a function defined by $\alpha(A,B)=1$, for all $A,B\in\mathcal{H}_c(X)$. Clearly all the conditions of Theorem (3.3.5) satisfies by assuming $c=\frac{3}{4}$. Then T has a fixed point 0.

Example 3.3.11. Let $X=[0,1]$, $E=\mathbb{R}^2$ and $P=\{(x,y)\in E: x,y\geq 0\}$. Define $d: X\times X\to E$ by $d(x,y)=(|x-y|,\frac{1}{2}|x-y|)$, $x,y\in X$. Then (X,d) is a complete cone metric space with normal constant of P equal to 1. Define $T: X\to\mathcal{H}_c(X)$ by

$$T(x)=\begin{cases}\{0\} & \text{if } x\in[0,\frac{1}{2}];\\ [0,\frac{1}{2}(x-\frac{1}{2})^2] & \text{if } x\in(\frac{1}{2},1].\end{cases} \tag{3.3.39}$$

Define $\psi : [0,\infty) \to [0,\infty)$ by $\psi(t) = t$ for all $t \in [0,\infty)$. Suppose that $\alpha : \mathcal{H}_c(X)\times\mathcal{H}_c(X) \to \mathbb{R}$ be a function defined by $\alpha(A,B) = 1$, for all $A,B \in \mathcal{H}_c(X)$. Clearly all the conditions of Theorem (3.3.7) satisfies by assuming $c = 0.49$. Then T has a fixed point 0.

Example 3.3.12. Let $X = \{a_1, a_2, a_3, ...\}$ be a countable set, $E = (l^2, \|.\|_2)$ and $P = \{\{x_n\}_{n\geq 1} \in l^2 : x_n \geq 0 (\forall n \geq 1)\}$. Let $x_i = \{\frac{3^i}{n}\}_{n\geq 1}$ for all $i \geq 1$ and note that $x_i \in l^2 (i \geq 1)$. Define the map $d : X \times X \to P$ by

$$d(a_i, a_j) = |x_i - x_j| = \{\frac{|3^i - 3^j|}{n}\}_{n\geq 1}$$

Then we can easily see that (X,d) is a complete cone metric space with the normal constant of P is equal to 1. We define the multifunction $T : X \to \mathcal{H}_c(X)$ by $Ta_1 = \{a_1\}$ and $Ta_i = \{a_1, a_2, ..., a_{i-1}\}$ for all $i \geq 1$. Again we define $\psi : [0,\infty) \to [0,\infty)$ by $\psi(t) = t$ for all $t \in [0,\infty)$ and let $\alpha : \mathcal{H}_c(X) \times \mathcal{H}_c(X) \to \mathbb{R}$ be a function defined by $\alpha(A,B) = 1$, for all $A,B \in \mathcal{H}_c(X)$. Then we see that T is a triangular α-admissible map satisfying all the conditions of Theorem (3.3.9) by assuming $c = \frac{1}{3}$ with a fixed point $Ta_1 = a_1$.

[0]Some of the materials presented in this chapter are taken from our paper (see-[23]) entitled "Fixed point on $\alpha - \psi$ multivalued contractive mapping in cone metric space", Acta et Commentationes Universitatis Tartuensis de Mathematica, (To appear in vol. 20(1), June 2016).

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Chapter 4

Fixed points for a class of set valued mappings over a metric space endowed with a graph structure

4.1 Introduction

First important work concerning the existence of fixed points of contractive mappings in an ordered metric spaces was initiated by Ran and Reur-Ings([24]). Jachymski([16]) had opened a new technique by replacing the order structure on a given metric space by a graph structure in search of fixed point of a mapping over a metric space endowed with a graph. Following this concept, Abbas ([1]) proved some fixed point theorems for a class of power type graph contraction mappings over a graph structured metric spaces. Bojor ([8]) had been succeeded in proving fixed point theorems for different types of contractive mappings on such spaces. Bojor ([8]- [9]) had also been able to prove some fixed point theorems for different types contraction mapping on metric spaces equipped with a graph structure. These results on fixed point theory for a single valued and multivalued mappings over such metric spaces had further opened up through the works of Beg et.al.([5]-[6]).

Given a metric space (X, d), denote Δ, the diagonal of $X \times X$. Let G be a directed graph such that the set of edges $E(G)$ of G contains the diagonal Δ and set of vertices $V(G)$ of G coincides with X.

If $a, b \in V(G)$, then a path in G from a to b of length $r \in \mathbb{N}$ is a finite sequence $(a_n)_{n=0,1,2,\ldots,r}$ of vertices such that $a_0 = a, a_r = b$ and $(a_{i-1}, a_i) \in E(G)$ for $i = 1, 2, ..., r$.

Reversing the direction of edges of the graph G, we obtain G^{-1} and we write

$$E(G^{-1}) = \{(x, y) \in X \times X : (y, x) \in E(G)\}. \tag{4.1.1}$$

4.2 Preliminaries

Definition 4.2.1. ([16]) A mapping $F : X \to X$ is said to be a Banach G-contraction or simply G-contraction if f preserves edges of G, i.e.,

$$\forall x, y \in X, (x, y) \in E(G) \Rightarrow (fx, fy) \in E(G),$$

and f decreases weights of edges of G in the following way:

$$\exists \alpha \in (0, 1), \forall x, y \in X, (x, y) \in E(G) \Rightarrow d(fx, fy) \leq \alpha d(x, y).$$

Example 4.2.2. ([16]) Any constant function $f : X \to X$ is a Banach G-contraction since $E(G)$ contains all loops.

Definition 4.2.3. ([16]) A mapping $f : X \to X$ is called orbitally continuous if for all $x, y \in X$ and any sequence $(k_n)_{n\in\mathbb{N}}$ of positive integers,

$$f^{k_n}x \to y \;\; implies \;\; f(f^{k_n}x) \to fy \;\; as \;\; n \to \infty.$$

Definition 4.2.4. A mapping $f : X \to X$ is called G-continuous if given $x \in X$ and a sequence $(x_n)_{n\in\mathbb{N}}$,

$$x_n \to x \;\; and \;\; (x_n, x_{n+1}) \in E(G) \;\; for \;\; n \in \mathbb{N} \;\; imply \;\; fx_n \to fx.$$

Definition 4.2.5. ([16]) A mapping $f : X \to X$ is called orbitally G-continuous if for all $x, y \in X$ and any sequence $(k_n)_{n\in\mathbb{N}}$ of positive integers,

$$f^{k_n}x \to y \;\; and \;\; (f^{k_n}x, f^{k_{n+1}}x) \in E(G) \;\; for \;\; n \in \mathbb{N} \;\; imply \;\; f(f^{k_n}x) \to fy.$$

Clearly, we have the following relations:

$$continuity \Rightarrow orbital \;\; continuity \Rightarrow orbital \;\; G - continuity;$$

$$continuity \Rightarrow G - continuity \Rightarrow orbital \;\; G - continuity.$$

If x and y are vertices in a graph G, then a path in G from x to y of length $N (N \in \mathbb{N} \cup \{0\})$ is a sequence $(x_i)_{i=0}^{N}$ of $N+1$ vertices such that,

$$x_0 = x, x_N = y \;\; and \;\; (x_{i-1}, x_i) \in E(G) \;\; for \;\; i = 1, 2, ..., N.$$

Note that a graph G is connected if there is a path between any two of its vertices and it is weakly connected if the undirected graph $\tilde{G}$ obtained from G by ignoring the direction of edges is connected.
In a metric space (X, d), denote

$$P(X) = \{A \in \mathcal{P}(X) : A \neq \Phi\}; P_b(X) = \{A \in P(X) : A \;\; is \;\; bounded\};$$

$$P_{cl}(X) = \{A \in P(X) : A \;\; is \;\; closed\}; P_{cp}(X) = \{A \in P(X) : A \;\; is \;\; compact\}.$$

Let $D : P(X) \times P(X) \to \mathbb{R}^+ \cup \{\infty\}$ defined as $D(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$. The Pompeiu-Hausdorff ([10]) functional

$$H : P(X) \times P(X) \to \mathbb{R}^+ \cup \{\infty\},$$

is defined by

$$H(A, B) = \max\{\sup_{a \in A} D(a, B), \sup_{b \in B} D(A, b)\}.$$

The diameter generalized functional δ generated by d is given by

$$\delta : P(X) \times P(X) \to \mathbb{R}^+ \cup \{\infty\},$$

and it is defined by

$$\delta(A, B) = \sup\{d(a, b) : a \in A, b \in B\}.$$

In a metric space (X, d) and let $T : X \to P(X)$ be a multivalued operator, then $x \in X$ is said to a fixed point for T iff $x \in Tx$ and the set $Fix(T) = \{x \in X :$

$x \in Tx\}$ is called fixed point set of T and $SFix(T) = \{x \in X : \{x\} = Tx\}$ is called strict fixed point set of T.

Let the graph of T is denoted by $G(T)$ and we write $G(T) = \{(x, y) : y \in Tx\}$.

Definition 4.2.6. ([13]) Let $\mathcal{S}$ denotes the class of functions $\alpha : \mathbb{R}^+ \to [0, 1)$ which satisfies the condition $\alpha(t_n) \to 1$ implies $t_n \to 0$.

This type of functions is called Geraghty functions.

4.3 Main Results

Theorem 4.3.1. *Let (X, d) be a complete metric space. Let G be a directed graph such that for any sequence $\{x_n\}_{n\in\mathbb{N}}$ in (X, d, G) satisfying $\{x_n\} \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{k_n}\}_{k\in\mathbb{N}}$ of $\{x_n\}$ satisfying $(x_{k_n}, x) \in E(G)$. Let $T : X \to P_b(X)$ be a multi valued operator satisfying the following conditions:*

(i)

$$\delta(Tx, Ty) \leq \alpha\{\delta(x, Tx) + \delta(y, Ty)\} + \beta d(x, y) + \gamma \max\{\delta(x, Ty), \delta(y, Tx)\}, \text{ where } 2\alpha + \beta + 2\gamma < 1 \text{ and for all } (x, y) \in E(G). \tag{4.3.1}$$

(ii) For each $x \in X$ the set

$$\tilde{X}_T(x) = \{y \in Tx : (x, y) \in E(G) \quad \text{and} \quad \delta(x, Tx) \leq pd(x, y), \text{for some } p \in (1, \tfrac{1-\alpha-\beta-\gamma}{\alpha+\gamma})\} \neq \Phi. \tag{4.3.2}$$

Then,

(I) $Fix(T) = SFix(T) \neq \Phi$

(II) If we further assume that $x^*, y^* \in Fix(T) \Rightarrow (x^*, y^*) \in E(G)$ then $Fix(T) = SFix(T) = \{x^*\}$.

Proof. Let $x_0 \in X$, Since $\tilde{X}_T(x_0) \neq \Phi$, $\exists x_1 \in T(x_0)$ and $p \in (1, \frac{1-\alpha-\beta-\gamma}{\alpha+\gamma})$, such that $(x_0, x_1) \in E(G)$ and $\delta(x_0, Tx_0) \leq pd(x_0, x_1)$.
Then by (4.3.1) we have ,

$$\begin{aligned}&\delta(x_1, Tx_1) \leq \delta(Tx_0, Tx_1)\\ &\leq \alpha\{\delta(x_0, Tx_0) + \delta(x_1, Tx_1)\} + \beta d(x_0, x_1) + \gamma \max\{\delta(x_0, Tx_1), \delta(Tx_0, x_1)\}\end{aligned} \tag{4.3.3}$$

We now examine the following cases:
Case-I:

$$\max\{\delta(x_0, Tx_1), \delta(Tx_0, x_1)\} = \delta(x_0, Tx_1)$$

Then form(4.3.3) we get that,

$$\delta(x_1, Tx_1) \leq \alpha pd(x_0, x_1) + \alpha\delta(x_1, Tx_1) + \beta d(x_0, x_1) + \gamma d(x_0, x_1) + \gamma\delta(x_1, Tx_1)$$

$$\Rightarrow \delta(x_1, Tx_1) \leq (\frac{\alpha p + \beta + \gamma}{1 - \alpha - \gamma})d(x_0, x_1) \tag{4.3.4}$$

Otherwise, Case-II:

$$\max\{\delta(x_0, Tx_1), \delta(Tx_0, x_1)\} = \delta(Tx_0, x_1)$$

and we get,

$$\delta(x_1, Tx_1) \leq \left(\frac{\alpha p + \beta + \gamma + \gamma p}{1 - \alpha}\right) d(x_0, x_1) \tag{4.3.5}$$

Since, $\tilde{X}_T(x_1) \neq \Phi$ for $x_1 \in X$, $\exists x_2 \in T(x_1)$ and $p \in (1, \frac{1-\alpha-\beta-\gamma}{\alpha+\gamma})$ such that $(x_1, x_2) \in E(G)$ and $\delta(x_1, Tx_1) \leq pd(x_1, x_2)$. But $d(x_1, x_2) \leq \delta(x_1, Tx_1)$.
Thus

$$d(x_1, x_2) \leq \delta(x_1, Tx_1) \leq (\frac{\alpha p + \beta + \gamma}{1 - \alpha - \gamma})d(x_0, x_1)$$

or

$$d(x_1, x_2) \leq \delta(x_1, Tx_1) \leq (\frac{\alpha p + \beta + \gamma + \gamma p}{1 - \alpha})d(x_0, x_1)$$

Again by applying (4.3.1) we get that,

$$\begin{aligned}&\delta(x_2, Tx_2) \leq \delta(Tx_1, Tx_2)\\ &\leq \alpha\{\delta(x_1, Tx_1) + \delta(x_2, Tx_2)\} + \beta d(x_1, x_2) + \gamma \max\{\delta(x_1, Tx_2), \delta(Tx_1, x_2)\}\end{aligned} \tag{4.3.6}$$

Now suppose that,

$$\max\{\delta(x_1, Tx_2), \delta(Tx_1, x_2)\} = \delta(x_1, Tx_2)$$

Then form(4.3.6) we get that,

$$\delta(x_2, Tx_2) \leq \alpha\{\delta(x_1, Tx_1) + \delta(x_2, Tx_2)\} + \beta d(x_1, x_2) + \gamma\delta(x_1, Tx_2)$$
$$\leq \alpha\delta(x_1, Tx_1) + \alpha\delta(x_2, Tx_2) + \beta d(x_1, x_2) + \gamma d(x_1, x_2) + \gamma\delta(x_2, Tx_2)$$

So,

$$\delta(x_2, Tx_2) \leq (\frac{\alpha}{1-\alpha-\gamma})\delta(x_1, Tx_1) + (\frac{\beta+\gamma}{1-\alpha-\gamma})d(x_1, x_2) \tag{4.3.7}$$

If

$$\delta(x_1, Tx_1) \leq (\frac{\alpha p+\beta+\gamma}{1-\alpha-\gamma})d(x_0, x_1),$$

$$\delta(x_2, Tx_2) \leq (\frac{\alpha+\beta+\gamma}{1-\alpha-\gamma})(\frac{\alpha p+\beta+\gamma}{1-\alpha-\gamma})d(x_0, x_1). \tag{4.3.8}$$

Again if

$$\delta(x_1, Tx_1) \leq (\frac{\alpha p+\beta+\gamma+\gamma p}{1-\alpha})d(x_0, x_1)$$

$$\delta(x_2, Tx_2) \leq (\frac{\alpha+\beta+\gamma}{1-\alpha-\gamma})(\frac{\alpha p+\beta+\gamma+\gamma p}{1-\alpha})d(x_0, x_1) \tag{4.3.9}$$

When,

$$\max\{\delta(x_1, Tx_2), \delta(Tx_1, x_2)\} = \delta(Tx_1, x_2)$$

then we see that,

$$\delta(x_2, Tx_2) \leq (\frac{\alpha+\gamma}{1-\alpha})\delta(x_1, Tx_1) + (\frac{\beta+\gamma}{1-\alpha})d(x_1, x_2)$$

So,

$$\delta(x_2, Tx_2) \leq (\frac{\alpha+\beta+2\gamma}{1-\alpha})\delta(x_1, Tx_1) \tag{4.3.10}$$

If

$$\delta(x_1, Tx_1) \leq (\frac{\alpha p+\beta+\gamma}{1-\alpha-\gamma})d(x_0, x_1) \tag{4.3.11}$$

then

$$\delta(x_2, Tx_2) \leq (\frac{\alpha+\beta+2\gamma}{1-\alpha})(\frac{\alpha p+\beta+\gamma}{1-\alpha-\gamma})d(x_0, x_1) \tag{4.3.12}$$

and if

$$\delta(x_1, Tx_1) \leq (\frac{\alpha p+\beta+\gamma+\gamma p}{1-\alpha})d(x_0, x_1)$$

then

$$\delta(x_2, Tx_2) \leq (\frac{\alpha+\beta+2\gamma}{1-\alpha})(\frac{\alpha p+\beta+\gamma+\gamma p}{1-\alpha})d(x_0, x_1) \tag{4.3.13}$$

Since, $\tilde{X}_T(x_2) \neq \Phi$, $\exists x_3 \in T(x_2)$ and $p \in (1, \frac{1-\alpha-\beta-\gamma}{\alpha+\gamma})$ such that $(x_2, x_3) \in E(G)$ and $\delta(x_2, Tx_2) \leq pd(x_2, x_3)$. So $d(x_2, x_3) \leq \delta(x_2, Tx_2)$.
Considering all the cases (4.3.8),(4.3.9),(4.3.12),(4.3.13) we see the following

$$\begin{aligned}
&d(x_2, x_3) \leq \delta(x_2, Tx_2) \leq (\frac{\alpha+\beta+\gamma}{1-\alpha-\gamma})(\frac{\alpha p+\beta+\gamma}{1-\alpha-\gamma})d(x_0, x_1)\\
or,\quad &d(x_2, x_3) \leq \delta(x_2, Tx_2) \leq (\frac{\alpha+\beta+\gamma}{1-\alpha-\gamma})(\frac{\alpha p+\beta+\gamma+\gamma p}{1-\alpha})d(x_0, x_1)\\
or,\quad &d(x_2, x_3) \leq \delta(x_2, Tx_2) \leq (\frac{\alpha+\beta+2\gamma}{1-\alpha})(\frac{\alpha p+\beta+\gamma}{1-\alpha-\gamma})d(x_0, x_1)\\
or,\quad &d(x_2, x_3) \leq \delta(x_2, Tx_2) \leq (\frac{\alpha+\beta+2\gamma}{1-\alpha})(\frac{\alpha p+\beta+\gamma+\gamma p}{1-\alpha})d(x_0, x_1)
\end{aligned} \tag{4.3.14}$$

By similar arguments we see that one of the following inequalities holds.

$$\begin{aligned}
\delta(x_3, Tx_3) &\leq (\frac{\alpha+\beta+\gamma}{1-\alpha-\gamma})^2(\frac{\alpha p+\beta+\gamma}{1-\alpha-\gamma})d(x_0, x_1)\\
\delta(x_3, Tx_3) &\leq (\frac{\alpha+\beta+\gamma}{1-\alpha-\gamma})^2(\frac{\alpha p+\beta+\gamma+\gamma p}{1-\alpha})d(x_0, x_1)\\
\delta(x_3, Tx_3) &\leq (\frac{\alpha+\beta+\gamma}{1-\alpha-\gamma})(\frac{\alpha+\beta+2\gamma}{1-\alpha})(\frac{\alpha p+\beta+\gamma}{1-\alpha-\gamma})d(x_0, x_1)\\
\delta(x_3, Tx_3) &\leq (\frac{\alpha+\beta+\gamma}{1-\alpha-\gamma})(\frac{\alpha+\beta+2\gamma}{1-\alpha})(\frac{\alpha p+\beta+\gamma+\gamma p}{1-\alpha})d(x_0, x_1)\\
\delta(x_3, Tx_3) &\leq (\frac{\alpha+\beta+\gamma}{1-\alpha-\gamma})(\frac{\alpha p+\beta+\gamma}{1-\alpha-\gamma})(\frac{\alpha+\beta+2\gamma}{1-\alpha})d(x_0, x_1)\\
\delta(x_3, Tx_3) &\leq (\frac{\alpha+\beta+2\gamma}{1-\alpha})(\frac{\alpha+\beta+\gamma}{1-\alpha-\gamma})(\frac{\alpha p+\beta+\gamma+\gamma p}{1-\alpha})d(x_0, x_1)\\
\delta(x_3, Tx_3) &\leq (\frac{\alpha+\beta+2\gamma}{1-\alpha})^2(\frac{\alpha p+\beta+\gamma}{1-\alpha-\gamma})d(x_0, x_1)\\
\delta(x_3, Tx_3) &\leq (\frac{\alpha+\beta+2\gamma}{1-\alpha})^2(\frac{\alpha p+\beta+\gamma+\gamma p}{1-\alpha})d(x_0, x_1)
\end{aligned} \tag{4.3.15}$$

Let us take
$(\frac{\alpha+\beta+\gamma}{1-\alpha-\gamma}) = A$, $(\frac{\alpha+\beta+2\gamma}{1-\alpha}) = B$, $(\frac{\alpha p+\beta+\gamma}{1-\alpha-\gamma}) = C$ and $(\frac{\alpha p+\beta+\gamma+\gamma p}{1-\alpha}) = D$.
Clearly, $0 < A, B, C, D < 1$. So from(4.3.15) we get,

$$\begin{aligned}
&d(x_3, x_4) \leq \delta(x_3, Tx_3) \leq A^2Cd(x_0, x_1)\\
or,\quad &d(x_3, x_4) \leq \delta(x_3, Tx_3) \leq A^2Dd(x_0, x_1)\\
or,\quad &d(x_3, x_4) \leq \delta(x_3, Tx_3) \leq ABCd(x_0, x_1)\\
or,\quad &d(x_3, x_4) \leq \delta(x_3, Tx_3) \leq ABDd(x_0, x_1)\\
or,\quad &d(x_3, x_4) \leq \delta(x_3, Tx_3) \leq ABCd(x_0, x_1)\\
or,\quad &d(x_3, x_4) \leq \delta(x_3, Tx_3) \leq ABDd(x_0, x_1)\\
or,\quad &d(x_3, x_4) \leq \delta(x_3, Tx_3) \leq B^2Cd(x_0, x_1)\\
or,\quad &d(x_3, x_4) \leq \delta(x_3, Tx_3) \leq B^2Dd(x_0, x_1) \qquad (4.3.16)
\end{aligned}$$

Continuing in this way we obtain a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that

(a) $(x_n, x_{n+1}) \in E(G)$, for each $n \in \mathbb{N}$.

(b)

$$\begin{aligned}
&d(x_n, x_{n+1}) \leq A^{n-1}Cd(x_0, x_1)\\
or,\quad &d(x_n, x_{n+1}) \leq A^{n-1}Dd(x_0, x_1)\\
or,\quad &d(x_n, x_{n+1}) \leq A^iB^{n-i-1}Cd(x_0, x_1)\\
or,\quad &d(x_n, x_{n+1}) \leq B^{n-1}Dd(x_0, x_1)\\
or,\quad &d(x_n, x_{n+1}) \leq B^{n-1}Cd(x_0, x_1)\\
or,\quad &d(x_n, x_{n+1}) \leq B^jA^{n-j-1}Dd(x_0, x_1), \qquad (4.3.17)
\end{aligned}$$

where $1 \leq i, j < (n-1)$.

(c)

$$\begin{aligned}
&\delta(x_n, Tx_n) \leq A^{n-1}Cd(x_0, x_1)\\
or,\quad &\delta(x_n, Tx_n) \leq A^{n-1}Dd(x_0, x_1)\\
or,\quad &\delta(x_n, Tx_n) \leq A^iB^{n-i-1}Cd(x_0, x_1)\\
or,\quad &\delta(x_n, Tx_n) \leq B^{n-1}Dd(x_0, x_1)\\
or,\quad &\delta(x_n, Tx_n) \leq B^{n-1}Cd(x_0, x_1)\\
or,\quad &\delta(x_n, Tx_n) \leq B^jA^{n-j-1}Dd(x_0, x_1), \qquad (4.3.18)
\end{aligned}$$

where $1 \leq i, j < (n-1)$.

Then form (4.3.17), we get that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X. Since X is complete, $\{x_n\} \to x^* \in X$ as $n \to \infty$. So by the assumption there exists a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ such that $(x_{n_k}, x^*) \in E(G)$ for each $k \in \mathbb{N}$. Now,

$$\begin{aligned}
&\delta(x^*, Tx^*) \leq d(x^*, x_{n_{k+1}}) + \delta(x_{n_{k+1}}, Tx^*) \leq d(x^*, x_{n_{k+1}}) + \delta(Tx_{n_k}, Tx^*) \\
&\leq d(x^*, x_{n_{k+1}}) + \alpha\{\delta(x_{n_k}, Tx_{n_k}) + \delta(x^*, Tx^*)\} + \\
&\beta d(x_{n_k}, x^*) + \gamma \max\{\delta(x_{n_k}, Tx^*), \delta(x^*, Tx_{n_k})\} \qquad (4.3.19)
\end{aligned}$$

Now there arise two cases. When,

$$\max\{\delta(x_{n_k}, Tx^*), \delta(x^*, Tx_{n_k})\} = \delta(x_{n_k}, Tx^*),$$

we obtain that,

$$\begin{aligned}
& \delta(x^*, Tx^*) \leq d(x^*, x_{n_{k+1}}) + \alpha\{\delta(x_{n_k}, Tx_{n_k}) \\
+ \quad & \delta(x^*, Tx^*)\} + \beta d(x_{n_k}, x^*) + \gamma\delta(x_{n_k}, Tx^*) \\
& \leq d(x^*, x_{n_{k+1}}) + \alpha p d(x_{n_k}, x_{n_{k+1}}) + \alpha\delta(x^*, Tx^*) \\
+ \quad & \beta d(x_{n_k}, x^*) + \gamma d(x_{n_k}, x^*) + \gamma\delta(x^*, Tx^*) \qquad (4.3.20)
\end{aligned}$$

As $k \to \infty$ we get,

$$\begin{aligned}
& \delta(x^*, Tx^*) = 0 \Rightarrow \{x^*\} = \{Tx^*\} \\
& \Rightarrow x^* \in SFix(T) \Rightarrow SFix(T) \neq \Phi \qquad (4.3.21)
\end{aligned}$$

Again when,

$$\max\{\delta(x_{n_k}, Tx^*), \delta(x^*, Tx_{n_k})\} = \delta(x^*, Tx_{n_k})$$

Then we obtain that,

$$\begin{aligned}
\delta(x^*, Tx^*) \leq \quad & d(x^*, x_{n_{k+1}}) + \alpha\{\delta(x_{n_k}, Tx_{n_k}) + \delta(x^*, Tx^*)\} \\
& + \beta d(x_{n_k}, x^*) + \gamma\delta(x^*, Tx_{n_k}) \\
& \leq d(x^*, x_{n_{k+1}}) + \alpha p d(x_{n_k}, x_{n_{k+1}}) + \alpha\delta(x^*, Tx^*) + \beta d(x_{n_k}, x^*) \\
& + \gamma d(x_{n_k}, x^*) + \gamma\delta(x_{n_k}, Tx_{n_k}) \qquad (4.3.22)
\end{aligned}$$

As $k \to \infty$ we get,

$$\begin{aligned} &\delta(x^*, Tx^*) = 0 \Rightarrow \{x^*\} = \{Tx^*\} \\ &\Rightarrow x^* \in SFix(T) \Rightarrow SFix(T) \neq \Phi. \end{aligned} \tag{4.3.23}$$

So for all cases we see that $SFix(T) \neq \Phi$.

Now we shall prove that $Fix(T) = SFix(T)$.

But it is obvious that $SFix(T) \subset Fix(T)$. So we have to prove that $Fix(T) \subset SFix(T)$.

Let $x^* \in Fix(T) \Rightarrow x^* \in Tx^*$, because $\Delta \subset E(G)$.

Then we have that, $(x^*, x^*) \in E(G)$.

By taking $x = y = x^*$ we get that,

$$\begin{aligned} & \delta(Tx^*) = \delta(Tx^*, Tx^*) \leq \alpha\{\delta(x^*, Tx^*) + \delta(x^*, Tx^*)\} \\ + \quad & \beta d(x^*, x^*) + \gamma \max\{\delta(x^*, Tx^*), \delta(x^*, Tx^*)\} \\ & \leq (2\alpha + \gamma)\delta(x^*, Tx^*) \end{aligned} \tag{4.3.24}$$

As, $x^* \in Tx^*$, we get that

$$\delta(x^*, Tx^*) \leq \delta(Tx^*) \text{and consequently } \delta(Tx^*) \leq (2\alpha + \gamma)\delta(x^*, Tx^*) \tag{4.3.25}$$

Assume $card(Tx^*) > 1$. So, $\delta(Tx^*) > 0$.

So by (4.3.25) we get that $(2\alpha+\gamma) > 1$ a contradiction to the fact that $2\alpha+\beta+\gamma < 1$.

So $\delta(Tx^*) = 0$ i.e. $\{Tx^*\}$ contains only single point i.e. $Tx^* = \{x^*\}$ i.e. $x^* \in SFix(T)$ and hence $Fix(T) \subset SFix(T)$.

So, $Fix(T) = SFix(T) \neq \Phi$.

Suppose $x^*, y^* \in Fix(T) = SFix(T)$ with $x^* \neq y^*$. Then

(1) $x^* \in SFix(T) \Rightarrow \delta(x^*, Tx^*) = 0$.

(2) $y^* \in SFix(T) \Rightarrow \delta(y^*, Ty^*) = 0$.

(3) $(x^*, y^*) \in E(G)$

By (4.3.1), we get

$$\begin{aligned} & d(x^*, y^*) = \delta(Tx^*, Ty^*) \\ & \leq \alpha\{\delta(x^*, Tx^*) + \delta(y^*, Ty^*)\} + \beta d(x^*, y^*) + \gamma \max\{\delta(x^*, Ty^*), \delta(y^*, Tx^*)\} \\ & \leq (\beta + \gamma)d(x^*, y^*) \Rightarrow (\beta + \gamma) \geq 1, \end{aligned} \tag{4.3.26}$$

which contradicts $(\beta+\gamma)<1$. So, $Fix(T)=SFix(T)=\{x^*\}$. □

Remark 4.3.2. In Theorem(4.3.1) we see that the results in Rhoades([25]) holds for the following cases.

(i) If we put $\alpha=\gamma=0$ then the operator becomes a Banach([3]) operator for $\beta<1$.

(ii) If $\alpha=\beta=0$ then it is a Chettejea([11]) type operator for $\gamma<\frac{1}{2}$.

(iii) If $\beta=\gamma=0$ then it is a Kannan([18]) type operator for $\alpha<\frac{1}{2}$.

Example 4.3.3. We consider the set $X=\{(0,0),(0,1),(1,0),(1,1)\}\subset\mathbb{R}^2$ with a metric d on X defined by

$$d(x,y)=|x_1-x_2|+|y_1-y_2|, \qquad \text{for } (x_1,y_1),(x_2,y_2)\in\mathbb{R}^2. \tag{4.3.27}$$

Let us define $T:X\to P_b(X)$ by

$$Tx=\begin{cases}(0,0), & \text{for } x=(0,0),\\ (0,0), & \text{for } x=(0,1),\\ \{(0,0),(0,1)\}, & \text{for } x=(1,0),\\ \{(0,0),(0,1)\}, & \text{for } x=(1,1).\end{cases} \tag{4.3.28}$$

Let $E(G):=\{\{(0,0),(0,1)\},\{(1,0),(0,1)\},\{(1,1),(0,0)\}\}\cup\Delta$.
It is a routine verification that all the conditions in Theorem(4.3.1) are satisfied for $\alpha=.49,\beta=.01,\gamma=.01$.
So, $Fix(T)=SFix(T)=\{0\}$.

Definition 4.3.4. Let (X,d) be a complete metric space and G be a directed graph. Let $T:X\to P_b$ be a multivalued operator. Then T is said to be Geraghty-G-contraction if there exists $\alpha\in\mathcal{S}$ such that

$$\delta(Tx,Ty)\leq\alpha(d(x,y))\times(d(x,y))\text{for all } (x,y)\in E(G). \tag{4.3.29}$$

Theorem 4.3.5. *Let (X,d) be complete metric space. Let G be a directed graph such that (X,d,G) satisfies the following property: for any sequence $\{x_n\}_{n\in\mathbb{N}}\in X$ such that $\{x_n\}\to x$ as $n\to\infty$, there exists a subsequence $\{x_{k_n}\}_{k\in\mathbb{N}}$ of $\{x_n\}$ such that $(x_{k_n},x)\in E(G)$.*
Let $T:X\to P_b(X)$ be a multi valued operator satisfying the following conditions:

(i) T is a Geraghty-G-contraction

(ii) for each $x \in X$, the set

$$\begin{aligned}\tilde{X}_T(x) \quad &= \{y \in Tx : (x, y) \in E(G) and \\ &\delta(x, Tx) \leq pd(x, y), for\ some\ p > 1\} \neq \Phi \end{aligned} \tag{4.3.30}$$

Then,

(I) $Fix(T) = SFix(T) \neq \Phi$

(II) If we additionally suppose that $x^*, y^* \in Fix(T) \Rightarrow (x^*, y^*) \in E(G)$ then $Fix(T) = SFix(T) = \{x^*\}$.

Proof. Let $x_0 \in X$, Since, $\tilde{X}_T(x_0) \neq \Phi$, $\exists x_1 \in T(x_0)$, $p > 1$ such that $(x_0, x_1) \in E(G)$ and $\delta(x_0, Tx_0) \leq pd(x_0, x_1)$.
Then by property of T as stated in (i),

$$\begin{aligned}&\delta(x_1, Tx_1) \leq \delta(Tx_0, Tx_1) \\ &\leq \alpha(d(x_0, x_1)) \times d(x_0, x_1) < d(x_0, x_1)\end{aligned} \tag{4.3.31}$$

For $x_1 \in X$, since $\tilde{X}_T(x_1) \neq \Phi$, $\exists x_2 \in T(x_1)$, $p > 1$ such that $(x_1, x_2) \in E(G)$ satisfying $\delta(x_1, Tx_1) \leq pd(x_1, x_2)$ and by the definition of δ we get $d(x_1, x_2) \leq \delta(x_1, Tx_1)$.
So by (4.3.31) we have,

$$d(x_1, x_2) < d(x_0, x_1) \tag{4.3.32}$$

By using the condition (i) in regard to T we get,

$$\begin{aligned}&\delta(x_2, Tx_2) \leq \delta(Tx_1, Tx_2) \\ &\leq \alpha(d(x_1, x_2)) \times d(x_1, x_2) < d(x_1, x_2) < d(x_0, x_1)\end{aligned} \tag{4.3.33}$$

Since, $\tilde{X}_T(x_2) \neq \Phi$, $\exists x_3 \in T(x_2), p > 1$ such that $(x_2, x_3) \in E(G)$ and $\delta(x_2, Tx_2) \leq pd(x_2, x_3)$. So $d(x_2, x_3) \leq \delta(x_2, Tx_2) < d(x_0, x_1)$.
Thus $\delta(x_3, Tx_3) < d(x_0, x_1)$. Continuing in this way we get a sequence $\{x_n\}_{n \in \mathbb{N}}$ such that

(a) $(x_n, x_{n+1}) \in E(G)$, for each $n \in \mathbb{N}$.

(b)

$$d(x_n, x_{n+1}) < d(x_0, x_1) \text{for each } n \in \mathbb{N}. \tag{4.3.34}$$

(c)

$$\delta(x_n, x_{n+1}) < d(x_0, x_1) \text{for each } n \in \mathbb{N}. \tag{4.3.35}$$

As, $d(x_n, x_{n+1}) < d(x_{n-1}, x_n)$ we see that $\{d(x_n, x_{n+1})\}$ is a decreasing sequence of real numbers which is bounded below and hence the sequence $\{d(x_n, x_{n+1})\}$ is convergent. Let $\{d(x_n, x_{n+1})\} \to r > 0$ as $n \to \infty$.
Then,

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \alpha(d(x_{n-1}, x_n)) \times d(x_{n-1}, x_n) \\ \frac{d(x_n, x_{n+1})}{(d(x_{n-1}, x_n))} &\leq \alpha(d(x_{n-1}, x_n)) \end{aligned} \tag{4.3.36}$$

Taking $n \to \infty$ we get,

$$\lim_{n\to\infty} \alpha(d(x_{n-1}, x_n)) \geq 1. \tag{4.3.37}$$

and consequently

$$\lim_{n\to\infty} \alpha(d(x_{n-1}, x_n)) = 1. \tag{4.3.38}$$

So from the definition of Geraghty function we get that

$$\lim_{n\to\infty} (d(x_{n-1}, x_n)) = 0. \tag{4.3.39}$$

Thus $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. By completeness of X we get $\{x_n\} \to x^* \in X$ as $n \to \infty$. So by the hypothesis there exists a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ such that $(x_{n_k}, x^*) \in E(G)$ for each $k \in \mathbb{N}$.
We shall prove that $x^* \in SFix(T)$. We have

$$\begin{aligned} \delta(x^*, Tx^*) &\leq d(x^*, x_{n_{k+1}}) + \delta(x_{n_{k+1}}, Tx^*) \leq d(x^*, x_{n_{k+1}}) + \delta(Tx_{n_k}, Tx^*) \\ &\leq d(x^*, x_{n_{k+1}}) + \alpha(d(x_{n_k}, x^*)) \times (d(x_{n_k}, x^*)) \\ &< d(x^*, x_{n_{k+1}}) + d(x_{n_k}, x^*) \end{aligned} \tag{4.3.40}$$

As $k \to \infty$ we get $\delta(x^*, Tx^*) = 0$. Hence, $x^* \in SFix(T)$. So, $SFix(T) \neq \Phi$.
By similar argument as in Theorem(4.3.1), we can prove that $Fix(T) = SFix(T)$.
Suppose $x^*, y^* \in Fix(T) = SFix(T)$ with $x^* \neq y^*$. Then

(i) $x^* \in SFix(T) \Rightarrow \delta(x^*, Tx^*) = 0$.

(ii) $y^* \in SFix(T) \Rightarrow \delta(y^*, Ty^*) = 0$.

(iii) $(x^*, y^*) \in E(G)$

So,

$$d(x^*, y^*) = \delta(Tx^*, Ty^*) \leq \alpha(d(x^*, y^*)) \times (d(x^*, y^*)) < d(x^*, y^*) \tag{4.3.41}$$

which is a contradiction. Hence, $Fix(T) = SFix(T) = \{x^*\}$. □

We now cite a supporting example in favour of the Theorem(4.3.5)

Example 4.3.6. Let, $\alpha \in \mathcal{S}$ be a Geraghty function([2]) defined by,

$$\alpha(t) = \frac{1}{1+t} \text{ ,for all } t \geq 0 \tag{4.3.42}$$

Consider X and the operator $T : X \to P_b(X)$ as same as in the Example(4.3.3). Then we see that all the conditions of the Theorem(4.3.5) satisfies and we get $Fix(T) = SFix(T) = \{0\}$.

Theorem 4.3.7. *Let (X, d) be complete metric space. Let G be a directed graph and $T : X \to P_b(X)$ be a multi valued operator. Suppose that $f : X \to \mathbb{R}_+$ defined $f(x) = \delta(x, Tx)$ is a lower semi-continuous mapping. Suppose that the following conditions holds:*

(i) $\exists \alpha, \beta, \gamma \in \mathbb{R}$ *with* $\alpha \neq 0$ *and* $2\alpha + \beta + 2\gamma < 1$ *such that*

$$\delta(y, Ty) \leq \alpha\delta(x, Tx) + \beta d(x, y) + \gamma \max\{\delta(x, Ty), \delta(y, Tx)\},$$
$$\textit{for all} \qquad (x, y) \in E(G) \cap G(T).$$

(ii) For each $x \in X$ *the set*

$$\begin{aligned} \tilde{X}_T(x) \quad &= \{y \in Tx : (x, y) \in E(G) \textit{and} \\ &\delta(x, Tx) \leq pd(x, y), \textit{for some } p \in (1, \tfrac{1-\alpha-\beta-\gamma}{\alpha+\gamma})\} \neq \Phi. \end{aligned} \tag{4.3.43}$$

Then, $Fix(T) = SFix(T) \neq \Phi$.

Proof. Let $x_0 \in X$, Since $\tilde{X}_T(x_0) \neq \Phi$, $\exists x_1 \in T(x_0)$ where $p \in (1, \frac{1-\alpha-\beta-\gamma}{\alpha+\gamma})$, such that $(x_0, x_1) \in E(G)$ and $\delta(x_0, Tx_0) \leq pd(x_0, x_1)$. Since $x_1 \in T(x_0)$ we get $(x_0, x_1) \in E(G) \cap G(T)$. Letting $y = x_1, x = x_0$ we get,

$$\begin{aligned} &\delta(x_1, Tx_1) \leq \alpha\delta(x_0, Tx_0) + \beta d(x_0, x_1) + \gamma \max\{\delta(x_0, Tx_1), \delta(Tx_0, x_1)\} \\ &\leq \alpha pd(x_0, x_1) + \beta d(x_0, x_1) + \gamma \max\{\delta(x_0, Tx_1), \delta(Tx_0, x_1)\} \end{aligned} \tag{4.3.44}$$

Now, $\delta(x_0, Tx_1) = \sup_{p \in Tx_1} d(x_0, p)$
So we get,

$$d(x_0, p) \leq d(x_0, x_1) + d(x_1, p), \text{for all } p \in Tx_1. \tag{4.3.45}$$

Thus we obtain,

$$\begin{aligned} \sup_{p \in Tx_1} d(x_0, p) \quad &\leq d(x_0, x_1) + \sup_{p \in Tx_1} d(x_1, p) \\ &= d(x_0, x_1) + \delta(x_1, Tx_1) \end{aligned} \tag{4.3.46}$$

Also, $\delta(Tx_0, x_1) = \sup_{q \in Tx_0} d(q, x_1)$.
So we get,

$$d(q, x_1) \leq d(x_0, x_1) + d(x_0, q), \text{for all } q \in Tx_0, \tag{4.3.47}$$

and hence we obtain,

$$\begin{aligned} \sup_{q \in Tx_0} d(q, x_1) \quad &\leq d(x_0, x_1) + \sup_{q \in Tx_0} d(x_0, q) \\ &= d(x_0, x_1) + \delta(x_0, Tx_0) \end{aligned} \tag{4.3.48}$$

Let

$$\max\{\delta(x_0, Tx_1), \delta(Tx_0, x_1)\} = \delta(x_0, Tx_1)(say)$$

Then from (4.3.44) and (4.3.46) we obtain that,

$$\begin{aligned} \delta(x_1, Tx_1) \quad &\leq \quad \alpha pd(x_0, x_1) + \beta d(x_0, x_1) + \gamma\delta(x_0, Tx_1) \\ &\leq \quad \alpha pd(x_0, x_1) + \beta d(x_0, x_1) + \gamma\{d(x_0, x_1) + \delta(x_1, Tx_1)\} \\ &\leq \quad (\frac{\alpha p + \beta + \gamma}{1 - \gamma})d(x_0, x_1) \end{aligned} \tag{4.3.49}$$

Otherwise if,

$$\max\{\delta(x_0, Tx_1), \delta(Tx_0, x_1)\} = \delta(Tx_0, x_1)$$

then from (4.3.44) and (4.3.48) we obtain that,

$$\begin{aligned}\delta(x_1,Tx_1) &\leq \alpha p d(x_0,x_1)+\beta d(x_0,x_1)+\gamma\delta(Tx_0,x_1)\\ &\leq \alpha p d(x_0,x_1)+\beta d(x_0,x_1)+\gamma\{\delta(Tx_0,x_1)+d(x_0,x_1)\}\\ &\leq (\alpha p+\beta+\gamma+\gamma p)d(x_0,x_1) \qquad (4.3.50)\end{aligned}$$

For $x_1 \in X$, Since, $\tilde{X}_T(x_1) \neq \Phi$, $\exists x_2 \in T(x_1)$ and $p \in (1, \frac{1-\alpha-\beta-\gamma}{\alpha+\gamma})$ such that $(x_1, x_2) \in E(G)$ and $\delta(x_1, Tx_1) \leq pd(x_1, x_2)$. Thus $d(x_1, x_2) \leq \delta(x_1, Tx_1)$. But $x_2 \in Tx_1$, and so $(x_1, x_2) \in E(G) \cap G(T)$ and $d(x_1, x_2) \leq \delta(x_1, Tx_1)$.
Thus either,

$$d(x_1,x_2) \leq \delta(x_1,Tx_1) \leq (\frac{\alpha p+\beta+\gamma}{1-\gamma})d(x_0,x_1)$$

or

$$d(x_1,x_2) \leq \delta(x_1,Tx_1) \leq (\alpha p+\beta+\gamma+\gamma p)d(x_0,x_1).$$

Again by taking $y = x_2, x = x_1$, we get by routine calculation, either

$$\delta(x_2,Tx_2) \leq (\frac{\alpha p+\beta+\gamma}{1-\gamma})d(x_1,x_2)$$

or

$$\delta(x_2,Tx_2) \leq (\alpha p+\beta+\gamma+\gamma p)d(x_1,x_2).$$

Hence

$$\begin{aligned}&\delta(x_2,Tx_2) \leq (\frac{\alpha p+\beta+\gamma}{1-\gamma})^2 d(x_0,x_1)\\ \textit{or}\quad &\delta(x_2,Tx_2) \leq (\alpha p+\beta+\gamma+\gamma p)(\frac{\alpha p+\beta+\gamma}{1-\gamma})d(x_0,x_1)\\ \textit{or}\quad &\delta(x_2,Tx_2) \leq (\alpha p+\beta+\gamma+\gamma p)^2 d(x_0,x_1) \qquad (4.3.51)\end{aligned}$$

For $x_2 \in X$, Since $\tilde{X}_T(x_2) \neq \Phi$, we get that $\exists x_3 \in T(x_1)$ and $p \in (1, \frac{1-\alpha-\beta-\gamma}{\alpha+\gamma})$ such that $(x_2, x_3) \in E(G)$ and $\delta(x_2, Tx_2) \leq pd(x_2, x_3)$. Then $d(x_2, x_3) \leq \delta(x_2, Tx_2)$. But $x_3 \in Tx_2$, so $(x_2, x_3) \in E(G) \cap G(T)$ and $d(x_2, x_3) \leq \delta(x_2, Tx_2)$. Thus for all the cases we have,

$$\begin{aligned}&d(x_2,x_3) \leq \delta(x_2,Tx_2) \leq (\frac{\alpha p+\beta+\gamma}{1-\gamma})^2 d(x_0,x_1)\\ \textit{or}\quad &d(x_2,x_3) \leq \delta(x_2,Tx_2) \leq (\alpha p+\beta+\gamma+\gamma p)(\frac{\alpha p+\beta+\gamma}{1-\gamma})d(x_0,x_1)\\ \textit{or}\quad &d(x_2,x_3) \leq \delta(x_2,Tx_2) \leq (\alpha p+\beta+\gamma+\gamma p)^2 d(x_0,x_1) \qquad (4.3.52)\end{aligned}$$

Again by taking $y = x_3, x = x_2$, we have,

$$\begin{aligned}\delta(x_3, Tx_3) &\leq \alpha\delta(x_2, Tx_2) + \beta d(x_2, x_3) + \gamma \max\{\delta(x_2, Tx_3), \delta(Tx_2, x_3)\}\\ &\leq (\alpha p + \beta)d(x_2, x_3) + \gamma \max\{\delta(x_2, Tx_3), \delta(Tx_2, x_3)\} \qquad (4.3.53)\end{aligned}$$

Again suppose that

$$\max\{\delta(x_2, Tx_3), \delta(Tx_2, x_3)\} = \delta(x_2, Tx_3)$$

Then from (4.3.53), we get,

$$\begin{aligned}\delta(x_3, Tx_3) &\leq (\alpha p + \beta)d(x_2, x_3) + \gamma\delta(x_2, Tx_3)\\ \delta(x_3, Tx_3) &\leq (\frac{\alpha p + \beta + \gamma}{1 - \gamma})d(x_2, x_3) \qquad (4.3.54)\end{aligned}$$

When,

$$\max\{\delta(x_2, Tx_3), \delta(Tx_2, x_3)\} = \delta(Tx_2, x_3)$$

Then from (4.3.53) we get,

$$\begin{aligned}\delta(x_3, Tx_3) &\leq (\alpha p + \beta)d(x_2, x_3) + \gamma\delta(Tx_2, x_3)\\ \delta(x_3, Tx_3) &\leq (\alpha p + \beta + \gamma + \gamma p)d(x_2, x_3) \qquad (4.3.55)\end{aligned}$$

So we have either

$$\delta(x_3, Tx_3) \leq (\frac{\alpha p + \beta + \gamma}{1 - \gamma})d(x_2, x_3)$$

or

$$\delta(x_3, Tx_3) \leq (\alpha p + \beta + \gamma + \gamma p)d(x_2, x_3)$$

Thus for all the cases there may arise the following:

$$\begin{aligned}\delta(x_3, Tx_3) &\leq (\frac{\alpha p + \beta + \gamma}{1 - \gamma})^3 d(x_0, x_1)\\ \delta(x_3, Tx_3) &\leq (\alpha p + \beta + \gamma + \gamma p)^2(\frac{\alpha p + \beta + \gamma}{1 - \gamma})d(x_0, x_1)\\ \delta(x_3, Tx_3) &\leq (\alpha p + \beta + \gamma + \gamma p)(\frac{\alpha p + \beta + \gamma}{1 - \gamma})^2 d(x_0, x_1)\\ \delta(x_3, Tx_3) &\leq (\alpha p + \beta + \gamma + \gamma p)^3 d(x_0, x_1) \qquad (4.3.56)\end{aligned}$$

Continuing in this way we can obtain a sequence $\{x_n\}_{n\in\mathbb{N}}$ satisfying following properties:

(d) $(x_n, x_{n+1}) \in E(G) \cap G(T)$, for each $n \in \mathbb{N}$.

(e)

$$\begin{aligned} & d(x_n, Tx_n) \leq (\frac{\alpha p+\beta+\gamma}{1-\gamma})^n d(x_0, x_1), \\ or \quad & d(x_n, Tx_n) \leq (\alpha p+\beta+\gamma+\gamma p)^i (\frac{\alpha p+\beta+\gamma}{1-\gamma})^{n-i} d(x_0, x_1) 1 \leq i \leq n-1, \\ or \quad & d(x_n, Tx_{n+1}) \leq (\alpha p+\beta+\gamma+\gamma p)^n d(x_0, x_1). \qquad (4.3.57) \end{aligned}$$

(f)

$$\begin{aligned} & \delta(x_n, Tx_n) \leq (\frac{\alpha p+\beta+\gamma}{1-\gamma})^n d(x_0, x_1), \\ or \quad & \delta(x_n, Tx_n) \leq (\alpha p+\beta+\gamma+\gamma p)^i (\frac{\alpha p+\beta+\gamma}{1-\gamma})^{n-i} d(x_0, x_1) 1 \leq i \leq n-1, \\ or \quad & \delta(x_n, Tx_n) \leq (\alpha p+\beta+\gamma+\gamma p)^n d(x_0, x_1). \qquad (4.3.58) \end{aligned}$$

As $(\frac{\alpha p+\beta+\gamma}{1-\gamma}) < 1$ and $(\alpha p+\beta+\gamma+\gamma p) < 1$, we get $d(x_n, Tx_{n+1}) \to 0$ as $n \to \infty$. So $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X and by completeness of X, $\{x_n\} \to x^* \in X(say)$ as $n \to \infty$. Using the lower continuity of f we have

$$0 \leq f(x^*) \leq \liminf_{n\to\infty} f(x_n) = 0$$

Thus $f(x^*) = 0$ and hence, $\delta(x^*, Tx^*) = 0$ implying that $x^* \in SFix(T)$.
Let $x^* \in Fix(T)$. Then $(x^*, x^*) \in G(T)$. Hence, $(x^*, x^*) \in E(G) \cap G(T)$.
Let $x = y = x^*$. So,

$$\begin{aligned} & \delta(Tx^*) = \delta(Tx^*, Tx^*) = \delta(x^*, Tx^*) \\ & \leq \alpha d(x^*, Tx^*) + \beta d(x^*, x^*) + \gamma \max\{\delta(x^*, Tx^*), \delta(x^*, Tx^*)\} \\ and \quad so \quad & \delta(Tx^*) \leq (\alpha+\gamma)\delta(x^*, Tx^*) \qquad (4.3.59) \end{aligned}$$

Then from (4.3.59), we get that $(\alpha+\gamma) \geq 1$, a contradiction.
Thus $\delta(Tx^*) = 0$ i.e. $Tx^* = \{x^*\}$ and so $Fix(T) = SFix(T) = \{x^*\} \neq \Phi$. □

Remark 4.3.8. Example(4.3.3) supports Theorem (4.3.7).

[0]Some of the materials presented in this chapter appeared in our paper (see-[20]) entitled "Fixed point for some classes of set valued mappings on a metric space endowed with a graph", Romai Journal, V.11, no. 1(2015), 115-129.

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Chapter 5

Coincidence point for isotone mappings on a partial ordered metric space

5.1 Introduction

Introducing partial order on a metric space (X, d), Ran and Reurings([17]), Nieto and Rodriguez-Lopez([13]-[12]-[14]), Wang([19]-[20]) started a new avenue for searching fixed point of various non-linear operators and related results over such space. This encompassed a new way which could be applied for the existence of solutions of linear or non linear equations by transferring a part of the contractive properties of the non linear mapping into its monotonicity properties. The development became also very useful for solving in many applied problems namely, to the existence of solutions for matrix equations or ordinary differential equations and integral equations (see- ([1]-[2]-[3]-[6]-[16])). On the otherhand, in 2006 Bhaskar and Lakshmikantham([6]) had initiated the notion of coupled fixed point and thereby proved some theorems on fixed point of mappings under certain conditions. Lakshmikantham and Ćirić([6]) again improved their results in 2009 with the introduction of g-monotone property. They also established the existence of coupled coincidence point and common fixed point in these relatively new spaces namely partial ordered metric space. Subsequently, many researchers like Borcut and Berinde (see- ([5]-[7]-[8])) had extended those earlier results. Amongst these generalizations, Berinde([5]) considered the most general contractive condition. Recently, researcher like Wang([19]-[20]) had obtained some k-dimensional fixed point theorem for mixed monotone mappings by extending the corresponding previous results on coupled, triple and quadruple fixed point theorems. Apart from these, some authors like Ishak([10]) and Zhang([21]) had developed fixed point theorems for a monotone mapping in ordered metric spaces while Harjani and Sadarangani([9]) had considered a single variable mapping in place of monotone mapping in a partially ordered metric spaces. Motivated by these results on a partially ordered metric spaces, we have established fixed point theorem for k variable isotone mapping. Some coincidence point theorems for k variable isotone mappings of generalized contractive character have also been proved here in this chapter. In this connection we can recall a following theorem.

Theorem 5.1.1. *([17]) Let $(X, \preceq)$ be a partially ordered set such that every pair $x, y \in X$ has an upper and lower bound. Let (X, d) be also a complete metric space and $T : X \to X$ be a continuous monotone mapping. Suppose that the following conditions hold;*
(i) there exists $\alpha \in (0, 1)$ with

$$d(T(x), T(y)) \leq \alpha d(x, y) \text{ for all } x \preceq y.$$

(ii) there exists $x_0 \in X$ with $x_0 \preceq T(x_0)$ (or $x_0 \succeq T(x_0)$).

Then T has a unique fixed point $x^* \in X$ and for each $x \in X$,

$$\lim_{n\to\infty} T^n(x) = x^*.$$

In subsequent times Theorem (5.1.1) was further generalized by several authors like Agarwal ([1]), Beg ([2]-[3]), Bhaskar ([6]), Nieto ([12]-[13], O'Regan ([15]), Petruşel ([16], Shaddad ([18]). Recently Wang ([19]-[20]) had proved some results for isotone mappings on partially ordered metric spaces.

This chapter is the continuation of their work and on T- isotone mappings that are available in current literatures. Here we have shown the existence of fixed points and coincidence points in a partially ordered metric space for a T-isotone and G-isotone mappings having different types of contractive characters.

5.2 Preliminaries

Let X be a non-empty set. We denote $X \times X \times ... \times X$ (k-times) by X^k, where $k \in \mathbb{N}$. A partially ordered set $(X, \preceq)$ which is also a metric space (X, d) is called a partially ordered metric space and denoted by $(X, d, \preceq)$.

Definition 5.2.1. ([18]) A function $\psi : [0, \infty) \to [0, \infty)$ is called an altering distance function if the following properties are satisfied:

(1) ψ is continuous and non-decreasing.

(2) $\psi(t) = 0$ if and only if $t = 0$.

Let Ψ be the collection of all such altering distance functions ψ as defined in Definition (5.2.1).

Let Φ denotes the set of all functions $\phi : [0, \infty) \to [0, \infty)$, satisfying

(a) ϕ is strictly increasing function,

(b) ϕ is continuous function.

Definition 5.2.2. ([19]) Let $(X, \preceq)$ be a partially ordered set and d be a metric on X. We say that $(X, d, \preceq)$ is regular if the following conditions hold:

(i) if a non-decreasing sequence $\{x_n\}$ is such that $x_n \to x$, then $x_n \preceq x$ for all n,

(ii) if a non-increasing sequence $\{y_n\}$ is such that $y_n \to y$, then $y_n \succeq y$ for all n.

If $x, y \in X$ are comparable (i.e. $x \preceq y$ or $y \preceq x$ holds) then we say, $x \asymp y$. Let $\{A, B\}$ be a partition of the set $\wedge_k = \{1, 2, ..., k\}$, i.e. $A \cup B = \wedge_k$ and $A \cap B = \phi$, $\Omega_{A,B} = \{\sigma : \wedge_k \to \wedge_k : \sigma(A) \subset A \quad \text{and} \quad \sigma(B) \subset B\}$ and $\Omega'_{A,B} =$

$\{\sigma : \wedge_k \to \wedge_k : \sigma(A) \subset B \quad \text{and} \quad \sigma(B) \subset A\}$. Let, $\sigma_1, \sigma_2, ..., \sigma_k$ be k mappings from $\wedge_k$ into itself and let Υ be the k-tuple $(\sigma_1, \sigma_2, ..., \sigma_k)$.

Assume that $(X, \preceq)$ be a partially ordered set and d be a metric on X such that (X, d) be a metric space. We use the notation

$$x \preceq_i y = \begin{cases} x \preceq y, & \text{if } i \in A, \\ x \succeq y, & \text{if } i \in B. \end{cases} \tag{5.2.1}$$

Let $(y_1, y_2, ..., y_k), (v_1, v_2, ..., v_k) \in X^k$. The natural partial ordering in the product space X^k is

$$(y_1, y_2, ..., y_k) \preceq (v_1, v_2, ..., v_k) \Leftrightarrow y_i \preceq_i v_i$$

which will be denoted in the sequel, for convention, by $\preceq$, also. Clearly, $(X^k, \preceq)$ is a partially ordered set. Particularly, we denote by A the odd numbers in $\wedge_k$ and by B its even numbers. The mapping $\rho_k : X^k \times X^k \to [0, \infty)$, given by

$$\rho_k(Y, V) = \frac{1}{k}[d(y_1, v_1) + d(y_2, v_2) + ... + d(y_k, v_k)],$$

where $Y = (y_1, y_2, ..., y_k), V = (v_1, v_2, ..., v_k) \in X^k$, defines a metric on X^k.

If $Y_n = (y_1^n, y_2^n, ..., y_k^n), Y = (y_1, y_2, ..., y_k) \in X^k$ then it is easy to show that

$$\rho_k(Y_n, Y) \to 0(as \quad n \to \infty) \Leftrightarrow d(y_i^n, y_i) \to 0(as \quad n \to \infty)$$

Definition 5.2.3. ([19]) Let $(X, \preceq)$ be a partially ordered set and T be a self-mapping on X^k. We say that T has an isotone property if, for any $Y_1, Y_2 \in X^k$,

$$Y_1 \preceq Y_2 \Rightarrow T(Y_1) \preceq T(Y_2)$$

Remark 5.2.4. We see that if $k = 1$ in Definition(5.2.3), then T is a non-decreasing mapping.

Definition 5.2.5. ([19]) An element $Y \in X^k$ is called a fixed point of the mapping $T : X^k \to X^k$ if $T(Y) = Y$.

Definition 5.2.6. Two mappings $T : X^k \to X^k$ and $G : X^k \to X^k$ are said to be commutative if $TG(Y) = GT(Y)$, for all $Y \in X^k$.

Definition 5.2.7. ([20]) Let $(X, \preceq)$ be a partially ordered set and $T : X^k \to X^k$ and $G : X^k \to X^k$ be two mappings. We say that T is a G-isotone mapping if, for any $Y_1, Y_2 \in X^k$

$$G(Y_1) \preceq G(Y_2) \Rightarrow T(Y_1) \preceq T(Y_2)$$

Definition 5.2.8. ([20]) An element $Y \in X^k$ is called a coincidence point of the mappings $T : X^k \to X^k$ and $G : X^k \to X^k$ if $T(Y) = G(Y)$. Furthermore, if $T(Y) = G(Y) = Y$, then we say that Y is a common fixed point of T and G.

Lemma 5.2.9. *([19]) Let $(X, \preceq)$ be a partially ordered set and d be a metric on X. If $(X, d, \preceq)$ is regular, then $(X^k, \rho_k, \preceq)$ is also regular.*

Definition 5.2.10. ([20]) A point $(x_1, x_2, ..., x_k) \in X^k$ is said to be

(a) a coupled coincidence point if for $k = 2$, $F(x_1, x_2) = g(x_1)$ and $F(x_2, x_1) = g(x_2)$,

(b) a coupled common fixed point if for $k = 2$, $F(x_1, x_2) = g(x_1) = x_1$ and $F(x_2, x_1) = g(x_2) = x_2$

5.3 Main Results

Theorem 5.3.1. *Let $(X, d, \preceq)$ be a partially ordered complete metric space. Let $T : X^k \to X^k$ be an isotone mapping satisfies the following conditions:*

(1) there exist an altering distance function ψ; an upper semi-continuous function $\theta : [0, \infty) \to [0, \infty)$ and a lower semi-continuous function $\beta : [0, \infty) \to [0, \infty)$ such that, for all $Y, V \in X^k$ with $Y \succeq V$,

$$\psi(\rho_k(T(Y), T(V))) \leq \theta(\rho_k(Y, V)) - \beta(\rho_k(Y, V))$$

where $\theta(0) = \beta(0) = 0$ and $\psi(t) - \theta(t) + \beta(t) > 0$, for all $t > 0$ and ρ_k is defined by above.

(2) there exists a $z_0 \in X^k$ such that $z_0 \asymp T(z_0)$.

(3) T is continuous or $(X, d, \preceq)$ is regular.

Then T has a fixed point in X.

Proof. Let $Z_0 \in X^k$. Then by considering the Picard iteration associated to T we get the sequence $\{Z_n\} \subseteq X^k$ defined by $Z_{n+1} = T(Z_n)$ for $n \geq 0$. If $Z_{n_0+1} = Z_{n_0}$ for some $n_0 \geq 0$ then obviously Z_{n_0} be a fixed point of T. So we assume that $Z_{n+1} \neq Z_n$ for every $n \geq 0$.
Without loss of generality we may assume that $Z_0 \preceq T(Z_0)$ as $Z_0 \asymp T(Z_0)$. So $Z_0 \preceq Z_1$. Since T is an isotone mapping, we can get $\{Z_n\}$ is non-decreasing. Putting $Y = Z_n$ and $V = Z_{n-1}$ in Theorem (5.3.1), we obtain that,

$$\begin{aligned} & \psi(\rho_k(Z_{n+1}, Z_n)) = \psi(\rho_k(T(Z_n), T(Z_{n-1}))) \\ \leq \; & \theta(\rho_k(Z_n, Z_{n-1})) - \beta(\rho_k(Z_n, Z_{n-1})) \end{aligned} \tag{5.3.1}$$

But we have $\psi(t) - \theta(t) + \beta(t) > 0$, for all $t > 0$. So,

$$\begin{aligned} & \psi(\rho_k(Z_n, Z_{n-1})) - \theta(\rho_k(Z_n, Z_{n-1})) + \beta(\rho_k(Z_n, Z_{n-1})) > 0 \\ & i.e. \quad \psi(\rho_k(Z_n, Z_{n-1})) > \theta(\rho_k(Z_n, Z_{n-1})) - \beta(\rho_k(Z_n, Z_{n-1})) \end{aligned} \tag{5.3.2}$$

So,

$$\frac{\psi(\rho_k(Z_{n+1}, Z_n))}{\psi(\rho_k(Z_n, Z_{n-1}))} < \frac{\theta(\rho_k(Z_n, Z_{n-1})) - \beta(\rho_k(Z_n, Z_{n-1}))}{\psi(\rho_k(Z_n, Z_{n-1}))} < 1$$

Thus

$$\psi(\rho_k(Z_{n+1}, Z_n)) < \psi(\rho_k(Z_n, Z_{n-1}))$$

As ψ is an altering distance function we get,

$$\rho_k(Z_{n+1}, Z_n) < \rho_k(Z_n, Z_{n-1}), \forall n \geq 1$$

Let $\delta_n = \rho_n(Z_n, Z_{n-1})$. Then the sequence $\{\delta_n\}$ is monotonically decreasing and bounded below. Therefore there exists some $\delta \geq 0$ such that $\lim_{n\to\infty} \delta_n = \delta$. We shall prove that $\delta = 0$. If $\delta > 0$, then by using the property of ψ, θ, β and taking $n \to \infty$ in (5.3.1) we obtain that

$$\begin{aligned} & \psi(\delta) \leq \theta(\delta) - \beta(\delta) \\ i.e. \quad & \psi(\delta) - \theta(\delta) + \beta(\delta) \leq 0 \end{aligned} \tag{5.3.3}$$

which is a contradiction. So, $\delta = 0$ i.e. $\lim_{n\to\infty} \rho_n(Z_n, Z_{n-1}) = 0$.
Now we claim that $\{Z_n\}$ is a Cauchy sequence in X. If $\{Z_n\}$ is not Cauchy then

there exists $\epsilon > 0$ and we can find subsequences $\{Z_{m(t)}\}$ and $\{Z_{n(t)}\}$ of $\{Z_n\}$ such that $n(t)$ is the minimal in the sense that

$$n(t) > m(t) \geq t \qquad \textit{and} \qquad \rho_k(Z_{m(t)}, Z_{n(t)}) > \epsilon.$$

Therefore we get, $\rho_k(Z_{m(t)}, Z_{n(t)-1}) \leq \epsilon$. So,

$$\begin{aligned}
& \epsilon < \rho_k(Z_{m(t)}, Z_{n(t)}) \\
\leq\ & \rho_k(Z_{m(t)}, Z_{m(t)-1}) + \rho_k(Z_{m(t)-1}, Z_{n(t)-1}) + \rho_k(Z_{n(t)-1}, Z_{n(t)}) \\
\leq\ & \rho_k(Z_{m(t)}, Z_{m(t)-1}) + \rho_k(Z_{m(t)}, Z_{m(t)-1}) \\
& + \rho_k(Z_{m(t)}, Z_{n(t)-1}) + \rho_k(Z_{n(t)-1}, Z_{n(t)}) \\
\leq\ & 2\rho_k(Z_{m(t)}, Z_{m(t)-1}) + \epsilon + \rho_k(Z_{n(t)-1}, Z_{n(t)}) \qquad (5.3.4)
\end{aligned}$$

Letting $t \to \infty$ in the inequality (5.3.4) and using $\lim_{n\to\infty} \delta_n = 0$ we get,

$$\lim_{n\to\infty} \rho_k(Z_{m(t)}, Z_{n(t)}) = \rho_k(Z_{m(t)-1}, Z_{n(t)-1}) = \epsilon$$

Since, $m(t) < n(t)$ i.e. $Z_{m(t)-1} \preceq Z_{n(t)-1}$. Let $Y = Z_{n(t)-1}$ and $V = Z_{m(t)-1}$, then from equation (5.3.1) we get that,

$$\begin{aligned}
& \psi(\rho_k(Z_{n(t)}, Z_{m(t)})) = \psi(\rho_k(T(Z_{n(t)-1}), T(Z_{m(t)-1}))) \\
\leq\ & \theta(\rho_k(Z_{n(t)-1}, Z_{m(t)-1})) - \beta(\rho_k(Z_{n(t)-1}, Z_{m(t)-1})) \qquad (5.3.5)
\end{aligned}$$

Letting $t \to \infty$ we obtain that, $\psi(\epsilon) \leq \theta(\epsilon) - \beta(\epsilon)$, this is a contradiction. Therefore $\{Z_n\}$ is a Cauchy sequence in metric space $(X^k, \preceq, \rho_k)$. Since $(X, d, \preceq)$ is complete metric space then $(X^k, \rho_k, \preceq)$ is also complete. Therefore there exists a $\tilde{Z} \in X^k$ such that $\lim_{n\to\infty} Z_n = \tilde{Z}$.

Since $Z_{n+1} = T(Z_n)$ and T is continuous, we see that $\tilde{Z}$ is a fixed point of T i.e. $T(\tilde{Z}) = \tilde{Z}$.

Again if $(X, d, \preceq)$ is regular then we get that $(X^k, \rho_k, \preceq)$ is regular by Lemma (5.2.9). As $\{Z_n\}$ is a non-decreasing sequence that converges to $\tilde{Z}$, we have $Z_n \preceq \tilde{Z}$ for all $n \geq 0$.

From (5.3.1) we get and $\psi \geq 0$ then,

$$\begin{aligned}
& \psi(\rho_k(Z_{n+1}, T(\tilde{Z}))) = \psi(\rho_k(T(Z_n), T(\tilde{Z}))) \\
\leq\ & \theta(\rho_k(Z_n, \tilde{Z})) - \beta(\rho_k(Z_n, \tilde{Z})) \qquad (5.3.6)
\end{aligned}$$

Taking limit $n \to \infty$ in (5.3.6) we obtain that, $\psi(\rho_k(T(\tilde{Z}), \tilde{Z})) = 0$ i.e. $T(\tilde{Z}) = \tilde{Z}$. So, $\tilde{Z}$ be the fixed point of T. $\square$

Theorem 5.3.2. *Let $(X, d, \preceq)$ be a partially ordered complete metric space. Let $G : X^k \to X^k$ and $T : X^k \to X^k$ be a G-isotone mapping that satisfies the following conditions:*

(1) there exist an altering distance function ψ; an upper semi-continuous function $\theta : [0, \infty) \to [0, \infty)$ and a lower semi-continuous function $\beta : [0, \infty) \to [0, \infty)$ such that, for all $Y, V \in X^k$ with $G(Y) \succeq G(V)$,

$$\psi(\rho_k(T(Y), T(V))) \leq \theta(\rho_k(G(Y), G(V))) - \beta(\rho_k(G(Y), G(V)))$$

where $\theta(0) = \beta(0) = 0$ and $\psi(t) - \theta(t) + \beta(t) > 0$, for all $t > 0$ and ρ_k is defined as above and $T(X^k) \subset G(X^k)$

and suppose either the condition satisfies

(a) T is continuous, G is continuous and commutes with T or

(b) $(X, d, \preceq)$ is regular and $G(X^k)$ is closed.

If there exists $Y_0 \in X^k$ such that $G(Y_0) \asymp T(Y_0)$ then T and G have a coincidence point in X.

Proof. Since $T(X^k) \subset G(X^k)$, it follows that there exists $Y_1 \in X^k$ such that $G(Y_1) = T(Y_0)$.
In general, there exist $Y_n \in X^k$ such that $G(Y_{n+1}) = T(Y_n), n \geq 0$.
We note that,

$$Z_0 = G(Y_0), \qquad Z_{n+1} = G(Y_{n+1}) = T(Y_n)$$

It is obvious that, if $Z_{n+1} = Z_n$ for some $n \geq 0$, then there is nothing to prove. So, we assume that $Z_{n+1} \neq Z_n$ for all $n \geq 0$.
Since $G(Y_0) \asymp T(Y_0)$, without loss of generality, we assume that, $G(Y_0) \preceq T(Y_0)$ i.e. $Z_0 \preceq Z_1$.
Assume that $Z_{n-1} \preceq Z_n$ i.e. $G(Y_{n-1}) \preceq G(Y_n)$. Since T is G-isotone mapping. We get,

$$Z_n = T(Y_{n-1}) \preceq T(Y_n) = Z_{n+1}, \forall n \geq 0.$$

So, $\{Z_n\}$ is non-decreasing. Again we get that,

$$\begin{aligned}\psi(\rho_k(Z_{n+1}, Z_n)) &= \psi(\rho_k(T(Y_n), T(Y_{n-1}))) \\ &\leq \theta(\rho_k(G(Y_n), G(Y_{n-1}))) - \beta(\rho_k(G(Y_n), G(Y_{n-1}))) \\ &= \theta(\rho_k(Z_n, Z_{n-1})) - \beta(\rho_k(\rho_k(Z_n, Z_{n-1})) \qquad (5.3.7)\end{aligned}$$

But $\psi(t) - \theta(t) + \beta(t) > 0$, for all $t > 0$. So we have,

$$\psi(\rho_k(Z_n, Z_{n-1})) > \theta(\rho_k(Z_n, Z_{n-1})) - \beta(\rho_k(\rho_k(Z_n, Z_{n-1})) \qquad (5.3.8)$$

Then from (5.3.7) and (5.3.8) we obtain that,

$$\begin{aligned}&\frac{\psi(\rho_k(Z_{n+1}, Z_n))}{\psi(\rho_k(Z_n, Z_{n-1}))} < 1 \\ i.e. \quad &\psi(\rho_k(Z_{n+1}, Z_n)) < \psi(\rho_k(Z_n, Z_{n-1})) \qquad (5.3.9)\end{aligned}$$

Since ψ is an altering distance function then we get, $\rho_k(Z_{n+1}, Z_n) < \rho_k(Z_n, Z_{n-1})$, $\forall n \geq 1$. Let $\delta_n = \rho_k(Z_n, Z_{n-1})$, the sequence $\{\delta_n\}$ is monotonically decreasing and bounded below. Therefore there exist a $\delta \geq 0$ such that $\lim_{n\to\infty} \delta_n = \delta$. We claim that $\delta = 0$. If $\delta > 0$ then by taking $n \to \infty$ in (5.3.7) we get that, $\psi(\delta) - \theta(\delta) + \beta(\delta) \leq 0$, a contradiction to our assumption. So, $\delta = 0$ i.e. $\lim_{n\to\infty} \rho_k(Z_n, Z_{n-1}) = 0$. Now we shall prove that, $\{Z_n\}$ is a Cauchy sequence in X^k. If not, then $\exists \; \epsilon > 0$ and we can find subsequences $\{Z_{m(t)}\}$ and $\{Z_{n(t)}\}$ of $\{Z_n\}$ such that $n(t)$ is the minimal in the sense that

$$n(t) > m(t) \geq t \qquad and \qquad \rho_k(Z_{m(t)}, Z_{n(t)}) > \epsilon.$$

Therefore we get, $\rho_k(Z_{m(t)}, Z_{n(t)-1}) \leq \epsilon$. So,

$$\epsilon < \rho_k(Z_{m(t)}, Z_{n(t)}) \leq 2\rho_k(Z_{m(t)}, Z_{m(t)-1}) + \epsilon + \rho_k(Z_{n(t)-1}, Z_{n(t)}). \qquad (5.3.10)$$

Letting $t \to \infty$ in (5.3.10) and using $\lim_{n\to\infty} \delta_n = 0$ we get

$$\lim_{n\to\infty} \rho_k(Z_{m(t)}, Z_{n(t)}) = \lim_{n\to\infty} \rho_k(Z_{m(t)-1}, Z_{n(t)-1}) = \epsilon$$

Since $m(t) < n(t)$ i.e. $Z_{m(t)-1} \preceq Z_{n(t)-1}$. Let $Y = Z_{n(t)-1}$ and $V = Z_{m(t)-1}$, then from the first condition of the theorem we get that,

$$\begin{aligned}&\psi(\rho_k(Z_{n(t)}, Z_{m(t)})) = \psi(\rho_k(T(Y_{n(t)-1}), T(Y_{m(t)-1}))) \\ \leq\; &\theta(\rho_k(G(Y_{n(t)-1}), G(Y_{m(t)-1}))) - \beta(\rho_k(G(Y_{n(t)-1}), G(Y_{m(t)-1}))) \qquad (5.3.11) \\ =\; &\theta(\rho_k(Z_{n(t)-1}, Z_{m(t)-1})) - \beta(\rho_k(Z_{n(t)-1}, Z_{m(t)-1})) \qquad (5.3.12)\end{aligned}$$

Letting $t \to \infty$ in (5.3.12) we have, $\psi(\epsilon) \leq \theta(\epsilon) - \beta(\epsilon)$, a contradiction as $\epsilon > 0$. So, $\{Z_n\}$ is a Cauchy sequence in $(X^k, \rho_k, \preceq_k)$. As $(X, d, \preceq)$ is complete then $(X^k, \rho_k, \preceq)$ is complete. Therefore $\exists\ \tilde{Z} \in X^k$ such that $\lim_{n\to\infty} Z_n = \tilde{Z}$ i.e. $\lim_{n\to\infty} G(Y_n) = \tilde{Z}$.

Suppose condition (a) of the Theorem (5.3.2) holds i.e. G is continuous. Then we have,

$$\lim_{n\to\infty} G(G(Y_{n+1})) = \lim_{n\to\infty} G(T(Y_n)) = \lim_{n\to\infty} T(G(Y_n))$$

i.e.

$$G(\tilde{Z}) = \lim_{n\to\infty} G(G(Y_{n+1})) = \lim_{n\to\infty} T(G(Y_n)) = T(\tilde{Z}),$$

which shows that $\tilde{Z}$ is a coincidence point of T and G.

Suppose condition (b) of the Theorem (5.3.2) holds i.e. $(X, d, \preceq)$ is regular. So $(X^k, \rho_k, \preceq)$ is regular by Lemma (5.2.9). As $\{Z_n\}$ is a non-decreasing sequence which converges to $\tilde{Z}$ then we get $Z_n \preceq \tilde{Z}$, for all n. Since $G(X^k)$ is closed and from condition (b) of the Theorem (5.3.2) we get $\tilde{Y} \in X^k$ for which $Y_n \to \tilde{Y}$. So

$$\lim_{n\to\infty} G(Y_{n+1}) = \lim_{n\to\infty} T(Y_n) = \tilde{Z} = G(\tilde{Y})$$

From the first condition of the Theorem we have,

$$\begin{aligned} & \psi(\rho_k(T(Y_n), T(\tilde{Y}))) \\ \leq\ & \theta(\rho_k(G(Y_n), G(\tilde{Y}))) - \beta(\rho_k(G(Y_n), G(\tilde{Y}))) \end{aligned} \tag{5.3.13}$$

Taking $n \to \infty$ in (5.3.13) we get,

$$\begin{aligned} & \lim_{n\to\infty} \psi(\rho_k(T(Y_n), T(\tilde{Y}))) = 0 \\ i.e.\quad & \psi(\rho_k(G(\tilde{Y}), T(\tilde{Y}))) = 0 \\ i.e.\quad & \rho_k(G(\tilde{Y}), T(\tilde{Y})) = 0 \\ i.e.\quad & G(\tilde{Y}) = T(\tilde{Y}) \end{aligned} \tag{5.3.14}$$

So, $\tilde{Y}$ is the coincidence point of T and G. □

Theorem 5.3.3. *Let $(X, d, \preceq)$ be a partially ordered complete metric space. Let $G : X^k \to X^k$ and $T : X^k \to X^k$ be an G-isotone mapping for which there exist $\phi \in \Phi$ and $\psi \in \Psi$ such that for all $Y, V \in X^k$ with $G(Y) \succeq G(V)$ satisfies the following:*

$$\phi(\rho_k(T(Y), T(V))) \leq \phi(\rho_k(G(Y), G(V))) - \psi(\rho_k(G(Y), G(V)))$$

where ρ_k is defined as above and $T(X^k) \subset G(X^k)$. Suppose that either

(1) T is continuous, G is continuous and commutes with T or

(2) $(X, d, \preceq)$ is regular and $G(X^k)$ is closed.

If there exists a $Y_0 \in X^k$ such that $G(Y_0) \asymp T(Y_0)$ then T and G have a coincidence point in X.

Proof. Since $T(X^k) \subset G(X^k)$, it follows that there exists $Y_1 \in X^k$ such that $G(Y_1) = T(Y_0)$.
In general, there exist $Y_n \in X^k$ such that $G(Y_{n+1}) = T(Y_n), n \geq 0$.
We note that,

$$Z_0 = G(Y_0), \qquad Z_{n+1} = G(Y_{n+1}) = T(Y_n)$$

It is obvious that, if $Z_{n+1} = Z_n$ for some $n \geq 0$ then there is nothing to prove. So we assume that $Z_{n+1} \neq Z_n$ for all $n \geq 0$.
Since $G(Y_0) \asymp T(Y_0)$, without loss of generality, we assume that, $G(Y_0) \preceq T(Y_0)$ i.e. $Z_0 \preceq Z_1$.
Assume that $Z_{n-1} \preceq Z_n$ i.e. $G(Y_{n-1}) \preceq G(Y_n)$. Since T is G-isotone mapping. We get,

$$Z_n = T(Y_{n-1}) \preceq T(Y_n) = Z_{n+1}, \forall n \geq 0.$$

So, $\{Z_n\}$ is non-decreasing. Again

$$\begin{aligned} \psi(\rho_k(Z_{n+1}, Z_n)) &= \psi(\rho_k(T(Y_n), T(Y_{n-1}))) \\ &\leq \psi(\rho_k(G(Y_n), G(Y_{n-1}))) - \phi(\rho_k(G(Y_n), G(Y_{n-1}))) \\ &= \psi(\rho_k(Z_n, Z_{n-1})) - \phi(\rho_k(\rho_k(Z_n, Z_{n-1})) \\ &< \psi(\rho_k(Z_n, Z_{n-1})) \end{aligned} \tag{5.3.15}$$

As ψ is a non decreasing function, so $\rho_k(Z_{n+1}, Z_n) < \rho_k(Z_n, Z_{n-1})$, $\forall n \geq 1$.
Let $\delta_n = \rho_k(Z_{n+1}, Z_n)$ is decreasing sequence and bounded below. Hence, $\{\delta_n\}$ is convergent. So there exist $\delta \geq 0$ such that $\lim\limits_{n\to\infty} \delta_n = \delta$. We shall now show that $\delta = 0$. Assume that $\delta > 0$. Then form (5.3.15) and taking $n \to \infty$ we obtain that,

$$\begin{aligned} &\lim_{n\to\infty} \psi(\delta_n) \leq \lim_{n\to\infty} \psi(\delta_{n-1}) - \lim_{n\to\infty} \phi(\delta_{n-1}) \\ i.e. \quad &\psi(\delta) \leq \psi(\delta) - \lim_{p\to\delta^+} \phi(p) \\ i.e. \quad &\psi(\delta) < \psi(\delta), \end{aligned} \tag{5.3.16}$$

which is a contraction. So, $\lim_{n\to\infty} \delta_n = 0$.
Again we claim that, $\{Z_n\}$ is a Cauchy sequence in X^k. If not, then $\exists\ \epsilon > 0$ and we can find subsequences $\{Z_{m(t)}\}$ and $\{Z_{n(t)}\}$ of $\{Z_n\}$ such that $n(t)$ is the minimal in the sense that

$$n(t) > m(t) \geq t \qquad \textit{and} \qquad \rho_k(Z_{m(t)}, Z_{n(t)}) > \epsilon.$$

Therefore we get, $\rho_k(Z_{m(t)}, Z_{n(t)-1}) \leq \epsilon$. So by routine verification we see that

$$\epsilon < \rho_k(Z_{m(t)}, Z_{n(t)}) \leq 2\rho_k(Z_{m(t)}, Z_{m(t)-1}) + \epsilon + \rho_k(Z_{n(t)-1}, Z_{n(t)}) \tag{5.3.17}$$

Letting $t \to \infty$ in (5.3.17) and using $\lim_{n\to\infty} \delta_n = 0$ we get,

$$\lim_{n\to\infty} \rho_k(Z_{m(t)}, Z_{n(t)}) = \lim_{n\to\infty} \rho_k(Z_{m(t)-1}, Z_{n(t)-1}) = \epsilon$$

Since $n(t) > m(t)$ we have, $Z_{m(t)-1} \preceq Z_{n(t)-1}$. Putting $Y = Z_{n(t)-1}$ and $V = Z_{m(t)-1}$ in Theorem (5.3.3) we have,

$$\begin{aligned} & \phi(\rho_k(Z_{n(t)}, Z_{m(t)})) = \phi(\rho_k(T(Y_{n(t)-1}), T(Y_{m(t)-1}))) \\ \leq\ & \phi(\rho_k(G(Y_{n(t)-1}), G(Y_{m(t)-1}))) - \psi(\rho_k(G(Y_{n(t)-1}), G(Y_{m(t)-1}))) \\ \leq\ & \phi(\rho_k(Z_{n(t)-1}, Z_{m(t)-1})) - \psi(\rho_k(Z_{n(t)-1}, Z_{m(t)-1})) \end{aligned} \tag{5.3.18}$$

Taking limit $t \to \infty$ in (5.3.18) we get,

$$\begin{aligned} \phi(\epsilon) &\leq \phi(\epsilon) - \lim_{t\to\infty} \psi(\rho_k(G(Y_{n(t)-1}), G(Y_{m(t)-1}))) \\ \phi(\epsilon) &\leq \phi(\epsilon) - \lim_{r_t\to\epsilon} \psi(r_t) < \phi(\epsilon), \end{aligned} \tag{5.3.19}$$

where $r_t = \rho_k(Z_{n(t)-1}, Z_{m(t)-1})$, a contradiction. Hence the sequence $\{Z_n\}$ is a Cauchy sequence in $(X^k, \rho_k, \preceq)$. As $(X, d, \preceq)$ is complete then $(X^k, \rho_k, \preceq)$ is complete. Therefore $\exists\ \tilde{Z} \in X^k$ such that, $\lim_{n\to\infty} Z_n = \tilde{Z}$ and we get, $\lim_{n\to\infty} G(Y_n) = \tilde{Z}$.
Now suppose that, condition (1) of the Theorem (5.3.3) satisfies i.e. G is continuous. Then we have $\lim_{n\to\infty} G(G(Y_{n+1})) = \tilde{Z}$. On the other hand by commutativity of T and G, we have

$$G(G(Y_{n+1})) = G(T(Y_n)) = T(G(Y_n)).$$

Taking limit as $n \to \infty$ we get,

$$G(\tilde{Z}) = \lim_{n\to\infty} G(G(Y_{n+1})) = \lim_{n\to\infty} T(G(Y_n)) = T(\tilde{Z}),$$

which shows that $\tilde{Z}$ is a coincidence point of T and G.
Again suppose that condition (2) holds in the Theorem (5.3.3) by assuming $(X, d, \preceq)$ is regular. So $(X^k, \rho_k, \leq)$ is regular by Lemma (5.2.9). As $\{Z_n\}$ is a non-decreasing sequence in X^k that converges to $\tilde{Z}$ in X^k, we get that $Z_n \preceq \tilde{Z}$ for all n. Since $G(X^k)$ is closed and from the condition (2) of Theorem (5.3.3), we get $\tilde{Y} \in X^k$ for which $Y_n \to \tilde{Y}$. So

$$\lim_{n\to\infty} G(Y_{n+1}) = \lim_{n\to\infty} T(Y_n) = \tilde{Z} = G(\tilde{Y})$$

By the condition of Theorem (5.3.3),

$$\begin{aligned} & \phi(\rho_k(T(Y_n), T(\tilde{Y}))) \\ \leq \ & \phi(\rho_k(G(Y_n), G(\tilde{Y}))) - \psi(\rho_k(G(Y_n), G(\tilde{Y}))) \end{aligned} \tag{5.3.20}$$

Taking $n \to \infty$ we get,

$$\begin{aligned} & \lim_{n\to\infty} \phi(\rho_k(T(Y_n), T(\tilde{Y}))) = 0 \\ i.e. \quad & \psi(\rho_k(G(\tilde{Y}), T(\tilde{Y}))) = 0 \\ i.e. \quad & \rho_k(G(\tilde{Y}), T(\tilde{Y})) = 0 \\ i.e. \quad & G(\tilde{Y}) = T(\tilde{Y}) \end{aligned} \tag{5.3.21}$$

So, $\tilde{Y}$ is the coincidence point of T and G. □

[0]Some of the materials presented in this chapter are taken from our paper (see-[4]) entitled "Coincidence point of isotone mappings in partially ordered metric spaces", (Communicated to).

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Chapter 6

Some results on random fixed points and its application

6.1 Introduction and Preliminaries

The study of random operator equations involving the method of functional analysis was first initiated by the Prague School of Probabilists led by A.Špaček ([47]) and Hanš([16]-[17]). Hanš ([16]-[17]) introduced a proper scheme of proving random fixed point theorems which has drawn a considerable interest for notable researchers like Adomian ([2]), Boyce ([11]), Bharucha-Reid ([9]-[10]), Kampe de Feriet([21]), Tsokos ([48]) and Tsokos-Padgete ([38]) and etc. Motivated by these works, researchers in fixed point theory have been attracted to prove analogues of basic deterministic fixed point theorems in stochastic version. This arises because of the significance of fixed point theorems in probabilistic functional analysis and probabilistic models with several applications. Issues relating to measurability of solutions, probabilistic and statistical aspect of random solution have arisen due to introduction of randomness. In a very recent paper of Saha and Debnath([41]), an appropriate scheme of iteration has also been presented in proving random fixed point theorems. Almost all random fixed point theorems are stochastic generalization of their classical deterministic counter parts. On the other hand Padgett ([37]) applied random fixed point theorem to prove the existence of a random solution of random non linear integral equation in a setting of Banach space. Achari ([37]), Saha and Dey ([42]), Saha and Ganguly ([44]) followed this and had been succeeded to develop the applications of random fixed point theory. In order to make the chapter self contained, we recall the basic definitions and preliminaries.

6.2 Preliminaries

Let (X, β_X) be a separable Banach space where β_X is a σ-algebra of Borel subsets of X, and let (Ω, β, μ) denote a complete probability measure space with measure μ, and β be a σ-algebra of subsets of Ω. For more details one can see Joshi and Bose ([20]).

Definition 6.2.1. ([20]) A mapping $x : \Omega \to X$ is said to be an X-valued random variable if the inverse image under the mapping x of every Borel set B of X belongs to β, that is $x^{-1}(B) \in \beta$ for all $B \in \beta_x$.

Definition 6.2.2. ([20]) A mapping $x : \Omega \to X$ is said to be a finite-valued random variable if it is constant on each finite number of disjoints sets $A_i \in \beta$ and is equal to 0 on $\Omega - (\cup_{i=1}^{n} A_i)$ and x is called a simple random variable if it is finitely valued and $\mu\{\omega : \|x(\omega)\| > 0\} < \infty$.

Definition 6.2.3. ([20]) A mapping $x : \Omega \to X$ is said to be a strong random variable if there exists a sequence $\{x_n(\omega)\}$ of simple random variables which converges to $x(\omega)$ almost surely i.e. there exists a set $A_0 \in \beta$ with $\mu(A_0) = 0$ such that

$$\lim_{n\to\infty} x_n(\omega) = x(\omega), \omega \in \Omega - A_0.$$

Definition 6.2.4. ([20]) A mapping $x : \Omega \to X$ is said to be a weak random variable if the function $x^*(x(\omega))$ is a real valued random variable for each $x^* \in X^*$, the space X^* denoting the first normed dual space of X.

In a separable Banach space X, the notions of strong and weak random variables $x : \Omega \to X$ (see-[20]) coincide, and in respect of such a space X, x is termed as a random variable.

Theorem 6.2.5. *([20]) Let $x, y : \Omega \to X$ be strong random variables and α, β be constants. Then the following statements hold:*
(i) $\alpha x(\omega) + \beta y(\omega)$ is a strong random variable. (ii) If $f(\omega)$ is a real valued random variable and $x(\omega)$ is a strong random variable, then $f(\omega)x(\omega)$ is a strong random variable. (iii) If $\{x_n(\omega)\}$ is a sequence of strong random variables converging strongly to $x(\omega)$ almost surely, i.e. if there exists a set $A_0 \in \beta$ with $\mu(A_0) = 0$ such that

$$\lim_{n\to\infty} \|x_n(\omega) - x(\omega)\| = 0$$

for every $\omega \notin A_0$, then $x(\omega)$ is a strong random variable.

Remark 6.2.6. ([20]) If X is a separable Banach space, then every strong and weak random variable is measurable.

Let Y be another Banach space.

Definition 6.2.7. ([20]) A mapping $F : \Omega \times X \to Y$ is said to be a random mapping if $F(\omega, x) = y(\omega)$ is a Y-valued random variable for every $x \in X$.

Definition 6.2.8. ([20]) A mapping $F : \Omega \times X \to Y$ is said to be a continuous random mapping if the set of all $\omega \in \Omega$ for which $F(\omega, x)$ is continuous function of x has measure one.

Definition 6.2.9. ([20]) A mapping $F : \Omega \times X \to Y$ is said to be demi-continuous at the $x \in X$, if $\|x_n - x\| \to 0$ as $n \to \infty$ implies $F(\omega, x_n) \underrightarrow{weakly} F(\omega, x)$ as $n \to \infty$ almost surely.

Definition 6.2.10. ([20]) A mapping $F : \Omega \times X \to Y$ is said to be demi-continuous random mapping where a Banach space Y is separable. Then, for any X-valued random variable x, the function $F(\omega, x(\omega))$ is a Y-valued random variable.

Remark 6.2.11. Since a continuous random mapping is demi-continuous, Theorem (6.2.5) is also true for continuous random mapping.

Definition 6.2.12. ([20]) An equation type $F(\omega, x(\omega)) = x(\omega)$, where $F : \Omega \times X \to X$ is a random mapping, is called a random fixed point equation.

Definition 6.2.13. ([20]) Any mapping $x : \Omega \to X$ which satisfies the random fixed point equation $F(\omega, x(\omega)) = x(\omega)$ almost surely is said to be a wide sense solution of the random fixed point equation.

Definition 6.2.14. ([20]) Any X-valued random variable $x(\omega)$ which satisfies $\mu\{\omega : F(\omega, x(\omega)) = x(\omega)\} = 1$ is said to be a random solution of the fixed point equation or a random fixed point of F.

Remark 6.2.15. ([20]) A random solution is a wide sense solution of the fixed point equation. But the converse is not true.

The following example supports our contention.

Example 6.2.16. ([20]) Let X be the set of all real numbers and let E be a non-measurable subset of X. Let $F : \Omega \times X \to Y$ be a random mapping defined as $F(\omega, x) = x^2 + x + 1$ for all $\omega \in \Omega$.
In case, the real-valued function $x(\omega) = 1$ for all $\omega \in \Omega$, is a random fixed point of F. However, the real-valued function $y(\omega)$ defined as

$$\begin{aligned} y(\omega) &= -1, \quad \omega \notin E \\ &= 1, \quad \omega \in E. \end{aligned}$$

is a wide sense solution of the fixed point equation $F(\omega, x(\omega)) = x(\omega)$ without being a random fixed point of F.

Theorem 6.2.17. *([20]) Let X be a separable Banach space and (Ω, β, μ) be a complete probability measure space. Let $T : \Omega \times X \to X$ be a continuous random operator satisfying*

$$\begin{aligned} \|T(\omega, x_1) - T(\omega, x_2)\| \leq\ & k_1(\omega)\left[\|x_1 - T(\omega, x_1)\| + \|x_2 - T(\omega, x_2)\|\right] \\ & + k_2(\omega)\left[\|x_1 - T(\omega, x_2)\| + \|x_2 - T(\omega, x_1)\|\right] \\ & + k_3 \|x_1 - x_2\| \end{aligned} \tag{6.2.1}$$

for all $\omega \in \Omega$ and $x_1, x_2 \in X$, $k_i(\omega) \geq 0$; $1 \leq i \leq 3$ are real valued random variables with $2k_1(\omega) + 2k_2(\omega) + k_3(\omega) < 1$ almost surely. Then there exists a unique random fixed point of T.

Remark 6.2.18. ([20]) *(I)* In the above theorem, setting $k_2(\omega) = k_3(\omega) = 0$, one can find random analogue of Kannan fixed point theorem ([22]) and in that case the operator $T : \Omega \times X \to X$ takes the form:

$$\|T(\omega, x_1) - T(\omega, x_2)\| \leq k_1(\omega)\left[\|x_1 - T(\omega, x_1)\| + \|x_2 - T(\omega, x_2)\|\right] \tag{6.2.2}$$

for all $\omega \in \Omega$ and $x_1, x_2 \in X$, $k_1(\omega) \geq 0$ is a real valued random variables with $k_1(\omega) < \frac{1}{2}$ almost surely.
(II) Setting $k_1(\omega) = k_3(\omega) = 0$, one can find random analogue of Chatterjea fixed point theorem ([15]) and in that case the operator $T : \Omega \times X \to X$ takes the form:

$$\|T(\omega, x_1) - T(\omega, x_2)\| \leq k_2(\omega)\left[\|x_1 - T(\omega, x_2)\| + \|x_2 - T(\omega, x_1)\|\right] \quad (6.2.3)$$

for all $\omega \in \Omega$ and $x_1, x_2 \in X$, $k_2(\omega) \geq 0$ is a real valued random variables with $k_2(\omega) < \frac{1}{2}$ almost surely.

Definition 6.2.19. ([43]) Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be a Ćirić operator if there are non-negative real valued functions q and δ over $X \times X$ satisfies

$$d(T^n(x), T^n(y) \leq q^n(x, y)\delta(x, y),$$

$$n = 1, 2, ... \text{ for all } x, y \in X \text{ where } q(x, y) < 1 \text{ with } \sup_{x,y \in X} q(x, y) = 1.$$

The set of all Ćirić operators be denoted by $Ci(X)$.

Theorem 6.2.20. *([43]) Let X be a complete metric space and $T \in Ci(X)$ satisfying*

$$\begin{aligned} d(T(x), T(y)) &\leq \alpha[d(x, T(x)) + d(y, T(y))] + \beta d(x, y) \\ &+\gamma \max\{d(x, T(y)), d(y, T(x))\} \end{aligned}$$

$\forall x, y \in X$, where $\alpha, \beta, \gamma \geq 0$ are such that $\max\{\alpha, \beta\} + \gamma < 1$. Then T has a unique fixed point in X.

Theorem 6.2.21. *([43]) Let X be a metric space and $T \in Ci(X)$ satisfying*

$$\begin{aligned} d(T(x), T(y)) &\leq \alpha[d(x, T(x)) + d(y, T(y))] + \beta d(x, y) \\ &+\gamma \max\{d(x, T(y)), d(y, T(x))\} \end{aligned}$$

$\forall x, y \in X$, where $\alpha, \beta, \gamma \geq 0$ are such that $\max\{\alpha, \beta\} + \gamma < 1$.
If the sequence of iterates $\{T^n(x_0)\}$ for some $x_0 \in X$ has a subsequence $\{T^{n_k}(x_0)\}$ with $\lim_k T^{n_k}(x_0) = u \in X$, then u is the unique fixed point of T and $\lim_n T^n(x_0) = u$.

Remark 6.2.22. Note that neither Kannan operator nor Chatterjea operator is continuous in general. But in case of random fixed point theorems, continuity of above two types of operators have been assumed.

6.3 Main Results

Definition 6.3.1. An operator $T : \Omega \times X \to X$ is said to be Ćirić random operator if there are two non-negative real valued functions q and δ over $X \times X$ satisfying $\|T^n(\omega, x_1) - T^n(\omega, x_2)\| \leq q^n(x_1, x_2).\delta(x_1, x_2),\ n = 1, 2, 3, \ldots$ for all $x, y \in X$ where $0 < q(x_1, x_2) < 1$ with $\sup\limits_{x_1, x_2 \in X} q(x_1, x_2) = 1$

Denote the class of random Ćirić operator by $Ci(\Omega \times X)$.

Theorem 6.3.2. *Let X be a sparable Banach space and (Ω, β, μ) be complete probability measure space and for each $\omega \in \Omega$, $T(\omega, x) \in Ci(\Omega \times X)$ and $T : \Omega \times X \to X$ be a continuous random operator such that for $\omega \in \Omega$, T satisfies*

$$\|T(\omega, x_1) - T(\omega, x_2)\| \leq \alpha(\omega)[\|x_1 - T(\omega, x_1)\| + \|x_2 - T(\omega, x_2)\|]$$
$$+\beta(\omega)\|x_1 - x_2\| + \gamma(\omega)\{\max[\|x_1 - T(\omega, x_2)\|, \|x_2 - T(\omega, x_1)\|]\} \quad (6.3.1)$$

$\forall x_1, x_2 \in X$, where $\alpha(\omega), \beta(\omega), \gamma(\omega)$ all are real valued random variables such that

$$\max\{\alpha(\omega), \beta(\omega)\} + \gamma(\omega) < 1.$$

Then T has a unique random fixed point in X.

Proof.

$$A = \{\omega \in \Omega : T(\omega, x) \text{ is continuous function of} X\}$$
$$B = \{\omega \in \Omega : \max\{\alpha(\omega), \beta(\omega)\} + \gamma(\omega) < 1\}$$

$$\begin{aligned} C_{x_1,x_2} &= \{\omega \in \Omega : \|T(\omega, x_1) - T(\omega, x_2)\| \\ &\leq \alpha(\omega)[\|x_1 - T(\omega, x_1)\| + \|x_2 - T(\omega, x_2)\|] + \beta(\omega)\|x_1 - x_2\| \\ &+\gamma(\omega)\{\max\{\|x_1 - T(\omega, x_2)\|, \|x_2 - T(\omega, x_1)\|\}\}\}. \end{aligned}$$

Let S be a countable dense subset of X. We now prove that

$$\bigcap_{x_1, x_2 \in X} C_{x_1,x_2} \cap A \cap B = \bigcap_{s_1, s_2 \in s} C_{s_1,s_2} \cap A \cap B.$$

Now for $s_1, s_2 \in S$,

$$\begin{aligned}\|T(\omega,s_1) &- T(\omega,s_2)\| \le \alpha(\omega)[\|s_1-T(\omega,s_1)\|+\|s_2-T(\omega,s_2)\|] \\ &+ \beta(\omega)\|s_1-s_2\|+\gamma(\omega)\{\max[\|s_1-T(\omega,s_2)\|,\|s_2-T(\omega,s_1)\|]\}.\end{aligned}$$

Let $x_1, x_2 \in X$. Since, S is dense in X, given $\delta_i(x_i) > 0$; $(i = 1, 2)$ there exist $s_1, s_2 \in S$ such that $\|x_i - s_i\| < \delta_i(x_i)$; $i = 1, 2$ such that for all $x_1, x_2 \in X$,

$$\begin{aligned}&\|s_1-T(\omega,s_1)\| \le \|s_1-x_1\|+\|x_1-T(\omega,x_1)\|+\|T(\omega,x_1)-T(\omega,s_1)\| \\ &\|s_1-T(\omega,s_2)\| \le \|s_1-x_1\|+\|x_1-T(\omega,x_2)\|+\|T(\omega,x_2)-T(\omega,s_2)\| \\ &\|s_2-T(\omega,s_1)\| \le \|s_2-x_2\|+\|x_2-T(\omega,x_1)\|+\|T(\omega,x_1)-T(\omega,s_1)\| \\ &\|s_2-T(\omega,s_2)\| \le \|s_2-x_2\|+\|x_2-T(\omega,x_2)\|+\|T(\omega,x_2)-T(\omega,s_2)\| \\ &\|s_1-s_2\| \le \|s_1-x_1\|+\|x_1-x_2\|+\|x_2-s_2\|\end{aligned} \tag{6.3.2}$$

We now examine the following cases:

Case-I

$$\begin{aligned}\|T(\omega,s_1)-T(\omega,s_2)\| \le\ & \alpha(\omega)[\|s_1-T(\omega,s_1)\|+\|s_2-T(\omega,s_2)\|] \\ &+\beta(\omega)\|s_1-s_2\|+\gamma(\omega)\|s_1-T(\omega,s_2)\|\end{aligned} \tag{6.3.3}$$

Case-II

$$\begin{aligned}\|T(\omega,s_1)-T(\omega,s_2)\| \le\ & \alpha(\omega)[\|s_1-T(\omega,s_1)\|+\|s_2-T(\omega,s_2)\|] \\ &+\beta(\omega)\|s_1-s_2\|+\gamma(\omega)\|s_2-T(\omega,s_1)\|\end{aligned} \tag{6.3.4}$$

Now we consider all the cases,
For case-I,

$$\begin{aligned}
\|T(\omega,x_1)-T(\omega,x_2)\| &\leq \|T(\omega,x_1)-T(\omega,s_1)\|+\|T(\omega,s_1)-T(\omega,s_2)\| \\
&\quad +\|T(\omega,s_2)-T(\omega,x_2)\| \\
&\leq \|T(\omega,x_1)-T(\omega,s_1)\|+\|T(\omega,s_2)-T(\omega,x_2)\| \\
&\quad +\alpha(\omega)[\|s_1-T(\omega,s_1)\|+\|s_2-T(\omega,s_2)\|] \\
&\quad +\beta(\omega)\|s_1-s_2\|+\gamma(\omega)\|s_1-T(\omega,s_2)\| \\
&\leq \|T(\omega,x_1)-T(\omega,s_1)\|+\|T(\omega,s_2)-T(\omega,x_2)\| \\
&\quad +\alpha(\omega)[\|s_1-x_1\|+\|x_1-T(\omega,x_1)\| \\
&\quad +\|T(\omega,x_1)-T(\omega,s_1)\|+\|s_2-x_2\| \\
&\quad +\|x_2-T(\omega,x_2)\|+\|T(\omega,x_2)-T(\omega,s_2)\|] \\
&\quad +\beta(\omega)[\|s_1-x_1\|+\|x_1-x_2\|+\|x_2-s_2\|] \\
&\quad +\gamma(\omega)[\|s_1-x_1\|+\|x_1-T(\omega,x_2)\| \\
&\quad +\|T(\omega,x_2)-T(\omega,s_2)\|] \\
&\qquad \text{(by using appropriate inequality in (6.3.2))} \\
&\leq [1+\alpha(\omega)]\|T(\omega,x_1)-T(\omega,s_1)\| \\
&\quad +[1+\alpha(\omega)+\gamma(\omega)]\|T(\omega,s_2)-T(\omega,x_2)\| \\
&\quad +[\alpha(\omega)+\beta(\omega)+\gamma(\omega)]\|s_1-x_1\| \\
&\quad +[\alpha(\omega)+\beta(\omega)]\|s_2-x_2\|+\alpha(\omega)\|x_1-T(\omega,x_1)\| \\
&\quad +\alpha(\omega)\|x_2-T(\omega,x_2)\|+\beta(\omega)\|x_1-x_2\| \\
&\quad +\gamma(\omega)\|x_1-T(\omega,x_2)\| \qquad (6.3.5)
\end{aligned}$$

Since for particular $\omega \in \Omega$, $T(\omega,x)$ is a continuous function of x, so for any $\epsilon > 0$ there exists $\delta_i > 0$ for $(i = i, 2)$ such that,

$$\|T(\omega,x_1)-T(\omega,s_1)\| < \epsilon/8 \quad \text{whenever} \quad \|x_1-s_1\| < \delta_1(x_1) \qquad (6.3.6)$$

and

$$\|T(\omega,x_2)-T(\omega,s_2)\| < \epsilon/8 \quad \text{whenever} \quad \|x_2-s_2\| < \delta_2(x_2) \qquad (6.3.7)$$

Now we choose $\delta_1 = \min(\delta_1(x_1), \epsilon/8)$ and $\delta_2 = \min(\delta_1(x_2), \epsilon/8)$
For such choice of δ_1, δ_2, we get form (6.3.5),

$$\begin{aligned}
&\|T(\omega, x_1) - T(\omega, x_2)\| \\
&\leq \epsilon/8 \times [1+\alpha(\omega)] + \epsilon/8 \times [1+\alpha(\omega)+\gamma(\omega)] + \epsilon/8 \times [\alpha(\omega)+\beta(\omega)+\gamma(\omega)] \\
&+\epsilon/8 \times [\alpha(\omega)+\beta(\omega)] + \alpha(\omega)\|x_1 - T(\omega, x_1)\| + \alpha(\omega)\|x_2 - T(\omega, x_2)\| \\
&+\beta(\omega)\|x_1 - x_2\| + \gamma(\omega)\|x_1 - T(\omega, x_2)\| \\
&\leq \epsilon/8 \times [1+\alpha(\omega)+1+\alpha(\omega)+\gamma(\omega)+\alpha(\omega)+\beta(\omega)+\gamma(\omega)+\alpha(\omega)+\beta(\omega)] \\
&+\alpha(\omega)\|x_1 - T(\omega, x_1)\| + \alpha(\omega)\|x_2 - T(\omega, x_2)\| \\
&+\beta(\omega)\|x_1 - x_2\| + \gamma(\omega)\|x_1 - T(\omega, x_2)\|
\end{aligned}$$

Since $\epsilon > 0$ be arbitrary, it follows that

$$\begin{aligned}
\|T(\omega, x_2) - T(\omega, s_2)\| \;\leq\; & \alpha(\omega)[\|x_1 - T(\omega, x_1)\| + \|x_2 - T(\omega, x_2)\|] \\
& +\beta(\omega)\|x_1 - x_2\| + \gamma(\omega)\|x_1 - T(\omega, x_2)\|
\end{aligned}$$

Similarly for case-II, we get

$$\begin{aligned}
\|T(\omega, x_1) - T(\omega, x_2)\| \;\leq\; & \alpha(\omega)[\|x_1 - T(\omega, x_1)\| + \|x_2 - T(\omega, x_2)\|] \\
& +\beta(\omega)\|x_1 - x_2\| + \gamma(\omega)\|x_2 - T(\omega, x_1)\|
\end{aligned}$$

Thus we get,

$$\begin{aligned}
&\|T(\omega, x_1) - T(\omega, x_2)\| \leq \alpha(\omega)[\|x_1 - T(\omega, x_1)\| + \|x_2 - T(\omega, x_2)\|] \\
&+\beta(\omega)\|x_1 - x_2\| + \gamma(\omega)\max[\|x_1 - T(\omega, x_2)\|, \|x_2 - T(\omega, x_1)\|]
\end{aligned}$$

Consequently,

$$\omega \in \bigcap_{x_1, x_2 \in X} C_{x_1, x_2}$$

$$\bigcap_{s_1, s_2 \in S} C_{s_1, s_2} \cap A \cap B \subset \bigcap_{x_1, x_2 \in X} C_{x_1, x_2} \cap A \cap B.$$

Similarly we can prove that

$$\bigcap_{x_1, x_2 \in X} C_{x_1, x_2} \cap A \cap B \subset \bigcap_{s_1, s_2 \in S} C_{s_1, s_2} \cap A \cap B. \tag{6.3.8}$$

So,

$$\bigcap_{x_1, x_2 \in X} C_{x_1, x_2} \cap A \cap B = \bigcap_{s_1, s_2 \in S} C_{s_1, s_2} \cap A \cap B. \tag{6.3.9}$$

Let,

$$N = \bigcap_{s_1, s_2 \in S} C_{s_1, s_2} \cap A \cap B. \tag{6.3.10}$$

Then, $\mu(N) = 1$. So for each $\omega \in N$, $T(\omega, x)$ is a deterministic operator due to Saha and Baisnab ([43]) and hence $T(\omega, x(\omega)) = x(\omega)$, where $x(\omega)$ is unique. $\square$

Theorem 6.3.3. *Let X be a sparable Banach space and (Ω, β, μ) be a complete probability measure space such that for each $\omega \in \Omega$, $T_j(\omega, x) \in Ci(\Omega \times X)$. Let $T : \Omega \times X \to X$ be a continuous random operator such that for each positive integer n*

$$\lim_{j \to \infty} T_j^n(\omega, x) = T^n(\omega, x)$$

for all $\omega \in \Omega$, T_j satisfies,

$$\begin{aligned} \|T_j(\omega, x_1) - T_j(\omega, x_2)\| \leq\ & \alpha(\omega)[\|x_1 - T_j(\omega, x_1)\| + \|x_2 - T_j(\omega, x_2)\|] \\ & + \beta(\omega)\|x_1 - x_2\| + \gamma(\omega)\{\max[\|x_1 - T_j(\omega, x_2)\|, \\ & \|x_2 - T_j(\omega, x_1)\|]\} \end{aligned} \tag{6.3.11}$$

$\forall x_1, x_2 \in X$, where $\alpha(\omega), \beta(\omega), \gamma(\omega)$ all are real valued random variables such that $\max\{\alpha(\omega), \beta(\omega)\} + \gamma(\omega) < 1$, possessing random fixed point u_j, $j = 1, 2, 3, \ldots$. Then T has the unique random fixed point u in X if and only if

$$u = \lim_{j \to \infty} u_j.$$

Proof. Given

$$T^n(\omega, x) = \lim_{j \to \infty} T_j^n(\omega, x), \forall x \in X,$$

where T_j has unique random fixed point u_j for $j = 1, 2, 3...$

Now for $x_1, x_2 \in X$,

$$\begin{aligned} \|T^n(\omega, x_1) - T^n(\omega, x_2)\| \leq\ & \|T^n(\omega, x_1) - T_j^n(\omega, x_1)\| + \|T_j^n(\omega, x_1) - T_j^n(\omega, x_2)\| \\ & + \|T_j^n(\omega, x_2) - T^n(\omega, x_2)\| \end{aligned} \tag{6.3.12}$$

Using the continuity of T and by routine calculation we see by (6.3.12) that $T(\omega, x) \in Ci(\Omega, X)$ satisfying,

$$\begin{aligned} & \|T(\omega, x_1) - T(\omega, x_2)\| \leq \alpha(\omega)[\|x_1 - T(\omega, x_1)\| + \|x_2 - T(\omega, x_2)\|] \\ & + \beta(\omega)\|x_1 - x_2\| + \gamma(\omega)\{\max[\|x_1 - T(\omega, x_2)\|, \|x_2 - T(\omega, x_1)\|]\}. \end{aligned}$$

Hence by Theorem(6.3.2) let u be the unique random fixed point of T.
Since $u_j = T_j(\omega, u_j)$ for $j = 1, 2, 3, ...$, we have

$$\begin{aligned} \|u - u_j\| &= \|T(\omega, u) - T_j(\omega, u_j)\| \\ &\leq \|T(\omega, u) - T_j(\omega, u)\| + \|T_j(\omega, u) - T_j(\omega, u_j)\| \\ &\leq \|T(\omega, u) - T_j(\omega, u)\| + \alpha(\omega)[\|u - T_j(\omega, u)\| + \|u_j - T_j(\omega, u_j)\|] \\ &+ \beta(\omega)\|u - u_j\| + \gamma(\omega)\{\max[\|u - T_j(\omega, u_j)\|, \|u_j - T_j(\omega, u)\|]\}. \end{aligned}$$

Taking $j \to \infty$ we can get that,
$\|u - u_j\| \leq \frac{1+\alpha+\gamma}{1-\beta-\gamma}\|T(\omega, u) - T_j(\omega, u_j)\| \to 0$. So,

$$u = \lim_{j\to\infty} u_j.$$

Conversely suppose that,

$$u = \lim_{j\to\infty} u_j.$$

Thus we get that,

$$\|T_j(\omega, u) - T_j(\omega, u_j)\| \leq \alpha(\omega)[\|u - T_j(\omega, u)\| + \|u_j - T_j(\omega, u_j)\|]$$
$$+\beta(\omega)\|u - u_j\| + \gamma(\omega)\{\max[\|u - T_j(\omega, u_j)\|, \|u_j - T_j(\omega, u)\|]\}. \quad (6.3.13)$$

By routine calculation and by taking $j \to \infty$ we get $\|u - T(\omega, u)\| = 0$. So u is the random fixed point for T and uniqueness follows immediately. □

6.4 Application

As the application of the Theorem (6.3.2), we can find a solution of a random nonlinear integral equation of the following form

$$x(t; \omega) = h(t; \omega) + \int_S k(t, s; \omega) f(s, x(s; \omega)) d\mu_0(s) \quad (6.4.1)$$

where

(i)$\mathcal{S}$ is a locally compact metric space with metric d on $S \times S$, μ_0 is a complete σ-finite measure defined on the collection of Borel subsets of S;

(ii) $\omega \in \Omega$, where ω is a supporting set of probability measure space (Ω, β, μ);

(iii) $x(t; \omega)$ is the unknown vector-valued random variables for each $t \in S$.

(iv) $h(t;\omega)$ is the stochastic free term defined for $t \in S$;

(v) $k(t,s;\omega)$ is the stochastic kernel defined for t and s in S and

(vi) $f(t,x)$ is vector-valued function of $t \in S$ and x and

(vi) $f(t,x)$ is a vector valued function of $t \in S$ and x.

The integral in Equation (6.4.1) is a Bochner integral. We will further assume that S is the union of a countable family of compact sets $\{C_n\}$ such that $C_1 \supset C_2 \supset ...$ and that for any other compact set S there is a C_i which contains it ([4]).

Definition 6.4.1. We define the space $C(S, L_2(\Omega,\beta,\mu))$ to be the space of all continuous functions from S into $L_2(\Omega,\beta,\mu)$ with the topology of uniform convergence on compacta i.e. for each fixed $t \in S$, $x(t;\omega)$ is a vector valued random variable such that

$$\|x(t;\omega)\|^2_{L_2(\Omega,\beta,\mu)} = \int_\Omega |x(t;\omega)|^2 \, d\mu(\omega) < \infty$$

It may be noted that $C(S, L_2(\Omega,\beta,\mu))$ is locally convex space (see- [49]) whose topology is defined by a countable family of semi-norms given by

$$\|x(t;\omega)\|_n = \sup_{t\in C_n} \|x(t;\omega)\|_{L_2(\Omega,\beta,\mu)}, n = 1, 2, ...$$

Moreover $C(S, L_2(\Omega,\beta,\mu))$ is complete relative to this topology, since $L_2(\Omega,\beta,\mu)$ is complete.

We further define $BC = BC(S, L_2(\Omega,\beta,\mu))$ to be the Banach space of all bounded continuous functions from S into $L_2(\Omega,\beta,\mu)$ with norm

$$\|x(t;\omega)\|_{BC} = \sup_{t\in S} \|x(t;\omega)\|_{L_2(\Omega,\beta,\mu)}$$

The space $BC \subset C$ is the space of all second order vector-valued stochastic process defined on S which are bounded and continuous in mean square.

We will consider the function $h(t;\omega)$ and $f(t, x(t;\omega))$ to be in the space $C(S, L_2(\Omega,\beta,\mu))$ with respect to the stochastic kernel. We assume that for each pair (t,s), $k(t,s;\omega) \in L_\infty(\Omega,\beta,\mu)$ and denote the norm by

$$\|k(t,s;\omega)\| = \|k(t,s;\omega)\|_{L_\infty(\Omega,\beta,\mu)} = \mu - ess \sup_{\omega\in\Omega} |k(t,s;\omega)|.$$

Also we suppose that $k(t,s;\omega)$ is such that $|||k(t,s;\omega)|||\,.\,\|x(s;\omega)\|_{L_2(\Omega,\beta,\mu)}$ is μ_0-integrable with respect to $s \in S$ for each $t \in S$ and $x(s;\omega)$ in $C(S, L_2(\Omega,\beta,\mu))$ and there exists a real valued function G defined μ_0-a.e. on S, so that $G(S)\,\|x(s;\omega)\|_{L_2(\Omega,\beta,\mu)}$ is μ_0-integrable and for each pair $(t,s) \in S \times S$,

$$||k(t,u;\omega) - k(s,u;\omega)||\,.\,\|x(u,\omega)\|_{L_2(\Omega,\beta,\mu)} \leq G(u)\,\|x(u,\omega)\|_{L_2(\Omega,\beta,\mu)}$$

μ_0-a.e. Further, for almost all $s \in S$, $k(t,s;\omega)$ will be continuous in t from S into $L_\infty(\Omega,\beta,\mu)$.

We now define the random integral operator T on $C(S, L_2(\Omega,\beta,\mu))$ by

$$(Tx)(t;\omega) = \int_S k(t,s;\omega)x(s;\omega)d\mu_0(s) \tag{6.4.2}$$

where the integral is a Bochner integral. Moreover, we have that for each $t \in S$, $(Tx)(t;\omega) \in L_2(\Omega,\beta,\mu)$ and that $(Tx)(t;\omega)$ is continuous in mean square by Lebesgue's Dominated Convergence Theorem. So $(Tx)(t;\omega) \in C(S, L_2(\Omega,\beta,\mu))$.

Definition 6.4.2. ([1], [34]) Let B and D be Banach spaces. The pair (B, D) is said to be admissible with respect to a random operator $T(\omega)$ if $T(\omega)(B) \subset D$.

Lemma 6.4.3. *([37]) The linear operator T defined by (6.4.1) is continuous from $C(S, L_2(\Omega,\beta,\mu))$ into itself.*

Lemma 6.4.4. *([37], [34]) If T is a continuous linear operator from $C(S, L_2(\Omega,\beta,\mu))$ into itself and $B, D \subset C(S, L_2(\Omega,\beta,\mu))$ are Banach spaces stronger than $C(S, L_2(\Omega,\beta,\mu))$ such that (B, D) is admissible with respect to T, then T is continuous from B into D.*

Remark 6.4.5. (see [37]) The operator T defined by (6.4.2) is a bounded linear operator from B into D.

It is to be noted that by a random solution of the equation (6.4.2), we will mean a function $x(t;\omega)$ in $C(S, L_2(\Omega,\beta,\mu))$ which satisfies the equation (6.4.2) μ_0-a.e.

We now state and prove the following theorem.

Theorem 6.4.6. *We consider the stochastic integral equation*

$$Ux(t;\omega) = h(t;\omega) + \int_S k(t,s;\omega)f(s, x(s;\omega))d\mu_0(s) \tag{6.4.3}$$

such that

(a)

$$\|U^n x_1(t;\omega) - U^n x_2(t;\omega)\| \leq q^n(x_1(t;\omega), x_2(t;\omega))\delta(x_1(t;\omega), x_2(t;\omega))$$

where $q(x_1(t;\omega), x_2(t;\omega)) < 1$ *with* $\sup q(x_1(t;\omega), x_2(t;\omega)) = 1$ *and* $\delta(x_1(t;\omega), x_2(t;\omega)) \geq 0$ *and* $x_1, x_2 \in C$.

(b) B *and* D *are Banach spaces stronger than* $C(S, L_2(\Omega, \beta, \mu))$ *such that* (B, D) *is admissible with respect to the integral operator defined by (6.4.1),*

(c) $x(t;\omega) \to f(t, x(t;\omega))$ *is a mapping from the set*

$$Q(\rho) = \{x(t;\omega) : x(t;\omega) \in D, \|x(t;\omega)\|_D \leq \rho\}$$

into the space B *satisfying*

$$\begin{aligned} &\|f(t, x_1(t;\omega)) - f(t, x_2(t;\omega))\|_B \leq \alpha(\omega)[\|x_1(t;\omega) - f(t, x_1(t;\omega))\|_D + \\ &\|x_2(t;\omega) - f(t, x_2(t;\omega))\|_D] + \beta(\omega)\|x_1(t,\omega) - x_2(t,\omega)\|_D + \\ &\gamma(\omega)[\max\{\|x_1(t;\omega) - f(t, x_2(t;\omega))\|_D, \|x_2(t;\omega) - f(t, x_1(t;\omega))\|_D]\}] \end{aligned} \quad (6.4.4)$$

for $x_1(t;\omega), x_2(t;\omega) \in Q(\rho)$, *where* $0 \leq \max\{\alpha(\omega), \beta(\omega)\} + \gamma(\omega) < 1$ *is a real valued random variable and* $h(t;\omega) \in D$.

Then there exits a unique random solution of (6.4.1) in $Q(\rho)$ *provided* $\frac{2\times l(\omega)}{1-\alpha(\omega)-\gamma(\omega)} < 1$ *and*

$$\|h(t,\omega)\|_D + \frac{2 + \max\{\alpha(\omega), \beta(\omega)\} \times l(\omega)}{1 - \alpha(\omega) - \gamma(\omega)}\|f(t,0)\|_B < \rho(1 - \frac{2l(\omega)}{1 - \alpha(\omega) - \gamma(\omega)})$$

where $l(\omega)$ *is the norm of* $T(\omega)$.

Proof. Define the operator U from $Q(\rho)$ into D by

$$(Ux)(t;\omega) = h(t,\omega) + \int_S k(t,s;\omega) f(s, x(s;\omega)) d\mu_0(s) \quad (6.4.5)$$

Now,

$$\begin{aligned} \|(Ux)(t;\omega)\|_D &\leq \|h(t,\omega)\|_D + l(\omega)\|f(t, x(t;\omega))\|_B \\ &\leq \|h(t,\omega)\|_D + l(\omega)\|f(t,0))\|_B + l(\omega)\|f(t, x(t,\omega)) - f(t,0)\|_B \end{aligned}$$

Then from (6.4.4) we get,

$$\|f(t,x(t;\omega))-f(t,0)\|_B \leq \alpha(\omega)[\|x(t;\omega)-f(t,x(t;\omega))\|_D+\|f(t,0)\|_D]+$$
$$\beta(\omega)\|x(t,\omega)\|_D+\gamma(\omega)[\max\{\|x(t;\omega)-f(t,0)\|_D,\|f(t,x(t;\omega))\|_D\}] \quad (6.4.6)$$

Suppose $\max\{\|x(t;\omega)-f(t,0)\|_D,\|f(t,x(t;\omega))\|_D\}=\|x(t;\omega)-f(t,0)\|_D$(say). Then by routine verification,

$$\begin{aligned}\|f(t,x(t;\omega))-f(t,0)\|_B &\leq \frac{\alpha(\omega)+\gamma(\omega)}{1-\alpha(\omega)}\|x(t;\omega)-f(t,x(t;\omega))\|_D+\\ &\frac{\alpha(\omega)}{1-\alpha(\omega)}\|f(t,0)\|_D+\frac{\beta(\omega)}{1-\alpha(\omega)}\|x(t;\omega)\|_D\end{aligned} \quad (6.4.7)$$

So we have,

$$\|f(t,x(t;\omega))-f(t,0)\|_B < \frac{1+\alpha(\omega)}{1-\alpha(\omega)}\|x(t;\omega)\|_D+\frac{1+\beta(\omega)}{1-\alpha(\omega)}\|f(t,0)\|_D \quad (6.4.8)$$

By routine calculation and by using (6.4.8) we obtain

$$\begin{aligned}\|(Ux)(t;\omega)\|_D &< \|h(t;\omega)\|_D+l(\omega)\|f(t,0)\|_B\\ &+l(\omega)\frac{1+\alpha(\omega)}{1-\alpha(\omega)}\rho+\frac{l(\omega)\times(1+\beta(\omega))}{1-\alpha(\omega)}\|f(t,0)\|_B\\ &<\rho\end{aligned} \quad (6.4.9)$$

Again suppose that, $\max\{\|x(t;\omega)-f(t,0)\|_D,\|f(t,x(t;\omega))\|_D\}=\|f(t,x(t;\omega))\|_D$. Then by routine verification we see that,

$$\begin{aligned}&\|f(t,x(t;\omega))-f(t,0)\|_B\\ &\leq \alpha(\omega)[\|x(t;\omega)-f(t,0)\|_D+\|f(t,0)-f(t,x(t;\omega))\|_D]+\alpha(\omega)\|f(t,0)\|_D\\ &+\beta(\omega)\|x(t,\omega)\|_D+\gamma(\omega)\|f(t,x(t;\omega))-f(t,0)\|_D+\gamma(\omega)\|f(t,0)\|_D\end{aligned} \quad (6.4.10)$$

Thus

$$\begin{aligned}&\|f(t,x(t;\omega))-f(t,0)\|_B \leq \frac{\alpha(\omega)}{1-\alpha(\omega)-\gamma(\omega)}\|x(t;\omega)-f(t,0)\|_D+\\ &\frac{\beta(\omega)}{1-\alpha(\omega)-\gamma(\omega)}\|x(t,\omega)\|_D+\frac{\alpha(\omega)+\gamma(\omega)}{1-\alpha(\omega)-\gamma(\omega)}\|f(t,0)\|_D\end{aligned} \quad (6.4.11)$$

Hence

$$\begin{aligned}\|f(t,x(t;\omega))-f(t,0)\|_B &< \frac{2\times\rho}{1-\alpha(\omega)-\gamma(\omega)}+\\ &\quad \frac{(1+\alpha(\omega))}{1-\alpha(\omega)-\gamma(\omega)}\|f(t,0)\|_D\end{aligned} \tag{6.4.12}$$

So

$$\begin{aligned}\|(Ux)(t;\omega)\|_D &< \|h(t;\omega)\|_D+l(\omega)\|f(t,0)\|_B\\ &\quad +l(\omega)\frac{1+\alpha(\omega)}{1-\alpha(\omega)}\rho+\frac{l(\omega)\times(1+\beta(\omega))}{1-\alpha(\omega)}\|f(t,0)\|_B\\ &<\rho\end{aligned} \tag{6.4.13}$$

Then by (6.4.9) and (6.4.13) we see that $(Ux)(t;\omega)\in Q(\rho)$ Again,

$$\begin{aligned}&\|(Ux_1)(t;\omega)-(Ux_2)(t;\omega)\|_D\\ &<\|\int_S k(t,s;\omega)[f(s,x_1(s;\omega))-f(s,x_2(s;\omega))]d\mu_0(s)\|_D\\ &\le l(\omega)\|f(t,x_1(t;\omega)-f(t,x_2(t;\omega)\|_B\\ &\le \alpha(\omega)\{\|x_1(t,\omega)-(Ux_1)(t,\omega)\|_D+\|x_2(t,\omega)-(Ux_2)(t,\omega)\|_D\}\\ &+\beta(\omega)\|x_1(t,\omega)-x_2(t,\omega)\|_D+\\ &\gamma(\omega)\{\max\{\|x_1-(Ux_2)(t,\omega)\|_D,\|x_2-(Ux_1)(t,\omega)\|_D\}\}\end{aligned} \tag{6.4.14}$$

Since, $\frac{l(\omega)}{1-\alpha(\omega)-\gamma(\omega)}<1$. Therefore $u(\omega)$ is a random contractive operator on $Q(\rho)$. Hence by Theorem (6.3.2) there exist a unique random fixed point $u(\omega)$ which is a random solution of (6.4.1). □

Example 6.4.7. Consider the following nonlinear stochastic Voltera integral equation:

$$x(t;\omega)=\int_0^t \frac{e^{-t-s}}{4(1+|x(s;\omega)|)}ds$$

where $t\in[0,1]$, $\omega\in[0,1]$ and $x\in C[0,1]$.

Also take $q(x_1,x_2)=\|x_1(t,\omega)\|$, $0<x_1(t;\omega)<1$ and $t,\omega\in[0,1]$, then $q(x_1,x_2)<1$ with $\sup\limits_{x_1,x_2\in C[0,1]} q(x_1,x_2)=1$.

Choose $\delta(x_1, x_2) = \|x_1(t;\omega) - x_2(t;\omega)\|$. So $\delta(x_1, x_2) > 0$.
Then we get,

$$\begin{aligned}\|U^n x_1(t;\omega) - U^n x_2(t;\omega)\| &< \frac{1}{n!}|t-0|^n.\|x_1 - x_2\| \\ &\leq q^n(x_1, x_2).\delta(x_1, x_2)\end{aligned}$$

Comparing with (6.4.2), we see that $h(t,\omega) = 0$,
$k(t,s;\omega) = \frac{1}{2}e^{-t-s}$, $f(s, x(s;\omega)) = \frac{1}{2(1+|x(s;\omega)|)}$.
Then one can check that equation (6.4.4) is satisfied with $\alpha(\omega) = 0$, $\beta(\omega) = \frac{1}{2}, \gamma(\omega) = 0$.

Comparing with integral operator equation (6.4.3), we see that the norm of $T(\omega)$ is $l(\omega) = \frac{1}{4}$ satisfying $\frac{l(\omega)}{1-\alpha(\omega)-\gamma(\omega)} < 1$ So, all the conditions of Theorem (6.4.6) are satisfied and hence there exists a random fixed point of the integral operator U satisfying (6.4.2).

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[0]Some of the materials presented in this chapter are taken from our paper (see-[33]) entitled "Random fixed point of *Ćirić* type contractive operator", Southeast Asian Bulletin of Mathematics, (accepted and to appear).

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Chapter 7

Fixed point analysis of a homogeneous isotropic turbulent flow

7.1 Introduction

In this chapter of Thesis, we would apply the mathematical analysis of the concept of fixed points to the physical turbulence problems of applied interest. In particular, we would concentrate our attention to the roles of fixed points to the behavior of the solutions of the problems of homogeneous isotropic turbulence in fluid mechanics. Here in the present case, we would deal with the consequences of the applications of fixed points to the well known equation of $Von - Kármán - Howarth$ ([30]) in homogeneous isotopic turbulence.

In 1948, Sir G. K. Batchelor([2]) studied the self-preserving features of the decaying-isotopic turbulence based on the assumption of invariance of Loitsiansky ([14]) integral. It was revealed later that such invariance of Loitsiansky ([14]) integral, is not completely valid.

So our analysis should concern primarily with the development of isotopic turbulence under different conditions.

Speziale and Bernard([26]-[27]) pointed out that generally two types of asymptotic solutions are obtainable for isotopic turbulence e.g., the turbulent kinetic energy decay as $K \sim t^{-1}$ and $K \sim t^{-\alpha}$, where $\alpha > 1$ with explicit dependence on the initial conditions. It is to be mentioned that by a fixed point analysis and numerical integration of the exact one point equations, the turbulent kinetic energy obeys the rule $K \sim t^{-1}$ which is consistent with asymptotic validity of high Reynolds number solutions.

In a recent paper, Chasnov([5]) commented that in three dimensions, high Reynolds number decay laws can be obtained by assuming that these law continue to depend on invariant or near invariant low wave numbers spectral coefficients but become independent of viscosity.

Further, the closure problems at high Reynolds numbers are yet unknown. Nevertheless Chasnov([5]) wrote that asymptotic decay laws of energy have been obtained by working in analogy to the final period of decay solutions. We discuss this point later on.

Accordingly, one assumes that scaling of the energy now depends on nonlinearly at the low wave number spectral coefficient and the time $'t'$ alone. Viscosity no longer enters the scaling law as an independent parameter. Chasnov([5]) mentioned by dimension analysis that with the usual turbulence phenomenology, the energy dissipation rate as a non zero limit with vanishing viscosity e.g. (see-[5]),

$$\langle u^2 \rangle = c_0 (4\pi B_0)^{\frac{2}{5}} t^{-\frac{6}{5}}, l \propto B_0^{\frac{1}{5}} t^{\frac{2}{5}} \tag{7.1.1}$$

$$\langle u^2 \rangle = c_2 (4\pi B_2)^{\frac{2}{7}} t^{-\frac{10}{7}}, l \propto B_0^{\frac{1}{7}} t^{\frac{2}{7}} \tag{7.1.2}$$

The result (7.1.1) is due to Saffman([23]) and the second result (7.1.2) was forwarded by Kolmogorov([10]). It is to be noted that the time dependence in (7.1.1) is expected to be exact because of invariance of B_0 and the explicit time-dependence is modified by the time-dependence of B_2.

In the present chapter we have included some additional decay laws of homogeneous and isotopic turbulence. That is, we have studied the process following Sen's([25]), a class of self preserving solutions of homogeneous and isotopic turbulence in respect of their relevance to fixed point analogy.

7.2 Homogeneous isotopic turbulence, Navier-Stokes equation and Von-K$\acute{a}$rm$\acute{a}$n-Howarth equation

To start with we consider two points x and x' at time t inside the homogeneous and isotopic turbulent flow field and accordingly,

$$\xi = x' - x, \qquad \xi_i\xi_i = r^2$$

$$\overline{u_i u'_j} = R_i^j = u'^2[-\frac{1}{2r}\frac{\partial f}{\partial r}\xi_i\xi_j + (f + \frac{1}{2}\frac{\partial f}{\partial r}\delta_{ij})]$$

where,

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{otherwise.} \end{cases} \tag{7.2.1}$$

$$u'^2 = \overline{u_1}^2 = \overline{u_2}^2 = \overline{u_3}^2,$$

$$u'^2 f(r) = (\overline{u_i u'_j})_{i=j,\xi_i=r}$$

$$u'^2 k(r) = (\overline{u_i u_j u'_k})_{i=j=k,\xi_i=r}$$

The most important equation([19]) of homogeneous isotopic turbulence at a point (x, t) can be written as,

$$\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \nu\nabla_x^2 u_i \tag{7.2.2}$$

$$\frac{\partial u_i}{\partial x_i} = 0 \tag{7.2.3}$$

where u_i be the turbulent velocity vector component, p be the fluctuating pressure and ν be the kinematic viscosity. Two point double and triple longitudinal velocity correlations denoted by $f(r, t)$ and $k(r, t)$ respectively and are defined in the standard way

$$f(r, t) = \frac{\overline{u(x, t)u(x + r, t)}}{\overline{u}^2} \tag{7.2.4}$$

$$k(r, t) = \frac{\overline{u^2(x, t)u(x + r, t)}}{(\overline{u}^2)^{3/2}} \tag{7.2.5}$$

where u is any component of turbulent velocity, x and $x+r$ are any two spatial points separated by a distance $r = \|\overrightarrow{r}\|$ in the direction of u.

$Von-K\acute{a}rm\acute{a}n-Howarth$([30]) derived the most important governing equation of homogeneous and isotopic turbulence as:

$$\frac{\partial(\overline{u^2 f})}{\partial t} = (\overline{u}^2)^{3/2}(\frac{\partial k}{\partial r} + \frac{4}{r}k) + 2\nu\overline{u}^2(\frac{\partial^2 f}{\partial r^2} + \frac{4}{r}\frac{\partial f}{\partial r}) \tag{7.2.6}$$

7.3 Some interesting results of turbulence

Now, we shall discuss about the solutions of this equation under different conditions and carry out the typical review of the analysis following Speziale and Bernard([27]).

We define turbulent kinetic energy $K = \frac{1}{2}\overline{u_i u_i}$ as a solution of the differential equation

$$\dot{K} = -\epsilon, \tag{7.3.1}$$

where, $\epsilon = \nu\overline{\omega_i\omega_i} \equiv \nu\omega^2$ is the turbulent dissipation rate, ω_i is the velocity vector and ω^2 is the enstrophy.

In addition to specifying K and ϵ we would consider one-point statistic at the initial time and specify $f(r)$ and $k(r)$ at any instant of time which satisfies $Von-K\acute{a}rm\acute{a}n-Howarth$ equation.

We shall now consider several important relations that were derived in the decay process of turbulence e.g.,

$$\frac{d\epsilon}{dt} = -2\nu\frac{\partial^3 S_{il,i}}{\partial r_l \partial r_j^2}(0) - 2\nu^2\frac{\partial^4 \mathcal{R}_{ii}}{\partial r_j^2 \partial r_l^2}(0) \tag{7.3.2}$$

where, $\mathcal{R}_{ii} = \overline{u}^2(3f + rf')$.

$$\frac{\partial^4 \mathcal{R}_{ii}}{\partial r_j^2 \partial r_l^2}(r) = \overline{u}^2[\frac{24}{r}f'''(r) + 11f^{iv}(r) + rf^{v}(r)] \tag{7.3.3}$$

The Taylor series expansion of $f'''(r)$ yields

$$f'''(r) = rf^{iv}(0) + \frac{r^3}{3!}f^{vi}(0) + ..., \tag{7.3.4}$$

$$\frac{\partial^4 \mathcal{R}_{ii}}{\partial r_j^2 \partial r_l^2}(0) = 35\overline{u}^2 f^{iv}(0), \tag{7.3.5}$$

and

$$\frac{\partial^3 S_{il,i}}{\partial r_l \partial r_j^2}(0) = \frac{35}{2} u_{rms}^3 k'''(0) \tag{7.3.6}$$

From the equation (7.3.2) we get,

$$\frac{d\epsilon}{dt} = -35\nu u_{rms}^3 k'''(0) - 70\nu^2 \overline{u}^2 f^{iv}(0) \tag{7.3.7}$$

where, $u_{rms} = \sqrt{2/3K}$. A form of (7.3.7) that is more useful for later analysis can be had by replacing $k'''(0)$ and $f^{iv}(0)$ by non dimensional parameters.

Skewness of the velocity derivative field

$$S_k = -\frac{\overline{(\frac{\partial u}{\partial x})^3}}{\overline{(\frac{\partial u}{\partial x})^2}^{3/2}} = -\lambda^3 [\frac{\partial^3 K}{\partial r^3}]_{r=0} \tag{7.3.8}$$

and the palenstrophy coefficient is given by,

$$G = \frac{\overline{(\frac{\partial^2 u}{\partial x^2})^2}}{\overline{(\frac{\partial u}{\partial x})^2}^2} = \lambda^4 [\frac{\partial^4 f}{\partial r^4}]_{r=0} \tag{7.3.9}$$

where, $R_T = \frac{K^2}{\nu\epsilon}$, $\lambda = [\frac{10\nu K}{\epsilon}]^{\frac{1}{2}}$ and we get, $\epsilon = -15\nu\overline{u^2}f''(0) = 15\nu\overline{(\frac{\partial u}{\partial x})^2}$.

The standard form of the ϵ-equation for homogeneous and isotopic turbulence can be derived as

$$\frac{d\epsilon}{dt} = S_K^* R_T^{\frac{1}{2}} \frac{\epsilon^2}{K} - G^* \frac{\epsilon^2}{K} \tag{7.3.10}$$

where, $S_K^* = \frac{7}{3\sqrt{15}} S_K$, $G^* = \frac{7}{15} G$ and $R_T = \frac{K^2}{\nu\epsilon}$. R_T is the dimensionless parameter that may be interpreted as a Reynolds number.

7.4 Similarity scales

It is important to mention that for isotopic turbulence to be self preserving in the sense of $Von - K\acute{a}rm\acute{a}n - Howarth$ and Batchelor([2]), we must express $f(r,t)$ and $k(r,t)$ as

$$f(r,t) = \tilde{f}(r/L), \qquad k(r,t) = \tilde{k}(r/L)$$

where, $L = L(t)$ is the unique specified similarity length scale.

For complete self preserving of all scales of turbulence, we define full range of $0 \leq r \leq \infty$ and restricted range of $0 \leq r \leq r_{max}$.

Following Batchelor([2]), we identify the relations:

$$\epsilon = -10\nu K[\frac{\partial^2 f}{\partial r^2}]_{r=0}, \qquad \lambda^2[\frac{\partial^2 f}{\partial r^2}]_{r=0} = -1.$$

Now, for complete self preserving isotopic turbulence

$$\frac{\lambda^2}{L^2}\tilde{f}''(0) = -1$$

from which it can be concluded that $L \propto \lambda$ (since $\tilde{f}''(0)$ is constant). For complete self preserving of isotopic turbulence we may set $L = \lambda$ and we get that, $S_K = -\tilde{K}'''(0)$=constant and $G = f^{iv}(0)$=constant, where a prime denote derivative with respect to the variable of $\frac{r}{\lambda}$. Consequently, $S_K = S_{K_0}$, $G = G_0$. This is a closed system for determining K and ϵ once K_0, ϵ_0, S_{K_0} and G_0 is provided. The assumption of complete self preservation is seen to lead to the closure assumption e.g. "if initial conditions for the skewness and the palinstrophy coefficient are provided in addition to initial conditions for K and ϵ then the energy decay can be calculated explicitly for all later times."

The $Von - K\acute{a}rm\acute{a}n - Howarth$ equation may now be put in the form

$$\begin{aligned} &10\tilde{f} + 2\eta^{-4}\frac{d}{d\eta}(\eta^4\frac{d\tilde{f}}{d\eta}) + \eta\frac{d\tilde{f}}{d\eta}(\frac{7}{3}G_0 - 5) \\ &= R_\lambda(\frac{7}{6}S_{K_0}\eta\frac{d\tilde{f}}{d\eta} - \eta^{-4}\frac{d(\eta^4\tilde{K})}{d\eta} \end{aligned} \tag{7.4.1}$$

where,

$$R_\lambda = (\overline{u^2})^{1/2}\frac{\lambda}{\nu} = (\frac{20}{3})^{1/2}R_t^{1/2}$$

Speziale and Bernard([27]) revisited the decay of turbulence problem and discussed that the similarity singularity scale ensure complete self preservation of full Von-K$\acute{a}$rm$\acute{a}$n-Howarth equation([30]). Using a fixed point analysis they demonstrated rigorously that the two asymptotic fully self preserving regimes of decay are possible.

7.5 Fixed point analysis

Zheng Ran([20]) presented in their report "large scale dynamics of isotopic turbulence" provided a table for comparison of self preserving regimes with fixed point analysis.

For carrying out fixed point analysis of

$$\frac{\partial(\overline{u^2 f})}{\partial t} = (\overline{u}^2)^{3/2}(\frac{\partial k}{\partial r} + \frac{4}{r}k) + 2\nu\overline{u}^2(\frac{\partial^2 f}{\partial r^2} + \frac{4}{r}\frac{\partial f}{\partial r}) \tag{7.5.1}$$

$$\dot{\epsilon} = \frac{7}{3\sqrt{15}} S_K \sqrt{R_T \frac{\epsilon^2}{K}} - \frac{7}{15} G \frac{\epsilon^2}{K}, \tag{7.5.2}$$

we combine them in to a single transformed equation for the turbulence Reynolds number R_t. Since,

$$\dot{R}_t = \frac{2k\dot{k}}{\nu\epsilon} - \frac{K^2}{\nu\epsilon^2}\dot{\epsilon} \tag{7.5.3}$$

it follows that

$$\dot{R}_t = -\frac{2K}{\nu} - \frac{7}{3\sqrt{15}} S_{K_0} \sqrt{R_t} \frac{K}{\nu} + \frac{7}{15} G_0 \frac{K}{\nu} \tag{7.5.4}$$

If the transformed dimensionless of time τ-defined by the relation

$$d\tau = \frac{\epsilon}{K} dt \tag{7.5.5}$$

is introduced into (7.5.4) we obtain the equation

$$\frac{dR_t}{d\tau} = R_t(\frac{7}{15}G_0 - 2 - \frac{7}{3\sqrt{15}} S_{K_0} \sqrt{R_t}) \tag{7.5.6}$$

The fixed points of (7.5.6) are obtained by setting $\frac{dR_t}{d\tau} = 0$ which yields the equation

$$R_{t_\infty}(\frac{7}{15}G_0 - 2 - \frac{7}{3\sqrt{15}} S_{K_0} \sqrt{R_{t_\infty}}) = 0, \tag{7.5.7}$$

where $(.)_\infty$ denotes the equilibrium value in the limit as $\tau \to \infty$.

The equation (7.5.7) has the solutions:

$$R_{t_\infty} = 0, \qquad \text{for } \tfrac{7}{15}G_0 \leq 2; \tag{7.5.8}$$

$$R_{t_\infty} = (\frac{\frac{7}{15}G_0 - 2}{\frac{7}{3\sqrt{15}S_{K_0}}})^2, \quad \text{for } \tfrac{7}{15}G_0 > 2. \tag{7.5.9}$$

It is also evident for (7.5.9) that in order to have an equilibrium high-Reynolds number isotopic flow field under self preserving condition, it is necessary that $G_0 \sim \sqrt{R_{t_\infty}}$.

By high Reynolds number isotopic turbulence we mean the case where $R_t >> 1$; for a low Reynolds number isotopic turbulence $R_t = O(1)$.

The fixed point $R_{t_\infty} = 0$ is associated with asymptotic solutions of K and ϵ that satisfy the differential equations

$$\dot{K} = -\epsilon \tag{7.5.10}$$

$$\dot{\epsilon} = -\frac{7}{15} G_0 \frac{\epsilon^2}{K} \tag{7.5.11}$$

The equations (7.5.10) and (7.5.11) have the exact solution

$$K = K_0 (1 + \frac{1}{\alpha} \frac{\epsilon_0 t}{K_0})^{-\alpha} \tag{7.5.12}$$

$$\epsilon = \epsilon_0 (1 + \frac{1}{\alpha} \frac{\epsilon_0 t}{K_0})^{-\alpha-1} \tag{7.5.13}$$

where $\alpha = \frac{1}{(\frac{7}{15} G_0 - 1)} > 1$ an exact asymptotic solution for K and ϵ may now be explained as:

$$K \sim t^{-\alpha}, \epsilon \sim t^{-\alpha+1} \tag{7.5.14}$$

we may now thought of a physically realizable range for $\frac{7}{15} G_0$ as less than 2.

7.6 Spectrum of homogeneous isotopic turbulence flow

Now we embark upon the spectral analysis of homogeneous isotopic turbulence with

$$E(k, t) = u^2(t) l(t) F(kl(t)),$$

where, $u^2(t)$ is the standard derivation of the characteristic velocity of the flow and it is related to turbulent kinetic energy $K(t)$ as $u^2(t) = \frac{2}{3} K(t)$, $l(t)$ is single length scale corresponding to velocity scale $u(t)$ and F is a time independent spectrum shape function.

We have the dynamical version of the energy spectrum $E(k,t)$ for isotopic turbulence

$$\frac{\partial E(k,t)}{\partial t} + 2\nu k^2 E(k,t) = T(k,t) \tag{7.6.1}$$

where $T(k,t)$ is the non-linear transfer term. This equation also may be put in the form

$$\frac{\partial}{\partial t}\int_0^k E(k)dk = -2(\nu + \nu'(k))\int_0^k k^2 E(k)dk \tag{7.6.2}$$

where $\nu'(k) = \mathcal{K}_H \int_k^\infty (\frac{E(k)}{k^3})^{1/2} dk$. We must mention that the equation (7.6.1) is equivalent to the Von-Kármán-Howarth equation([30])($Stanisić$).

We shall now consider a most general type of self similar solution of the equation (7.6.1) as forwarded by Sen([25]) with Heisenberg's form the energy transfer function $T(k,t)$.

Then we get that,

$$E(k,t) = \frac{1}{k^2}\frac{1}{k_0^3 t_0^2}(\frac{t_0}{t})^{(2-3c)} f(\frac{k}{k_0}\frac{t^c}{t_0^c}) \tag{7.6.3}$$

where c is a constant with $0 < c < \frac{2}{3}$. Let $x = (\frac{k}{k_0}\frac{t^c}{t_0^c})$ we find, important properties of the asymptotic solution $f(x)$ as:

$$\begin{aligned} f(x) &\sim x^{(\frac{2-3c}{c})} \qquad (x \to 0) \\ f(x) &\sim x^{(-\frac{5}{3})} \qquad (x \to \infty) \end{aligned} \tag{7.6.4}$$

We now calculate the turbulent energy by the formula

$$u'^2 = \int_0^\infty E(k,t)dk \tag{7.6.5}$$

by the relation

$$u'^2 = \frac{1}{k^2}\frac{1}{(k_0 t_0^c)^2} t^{2(c-1)} \int_0^\infty f(x)dx. \tag{7.6.6}$$

We can write this as,

$$u'^2 \sim t^{2(c-1)} \tag{7.6.7}$$

From Taylor's scale of turbulence([29]) we obtain that,

$$\lambda^2 = (\frac{5}{1-c})\nu t. \tag{7.6.8}$$

7.7 Discussion of results

We now construct the following two tables and put the numerical estimates of various characters of isotopic turbulent flows in it.

Two additional slower decay laws e.g., $u'^2 \sim t^{-1.36}$ and $u'^2 \sim t^{-1.43}$ that have been included in the present analysis satisfies all the criterion like the other power laws of tables and the cases have also been dealt by Chasnov([5]). It is to notice that the calculations carried out with $c = .285$ is very close to the case that corresponds to the case with $c = \frac{2}{7}$.

Table 7.1: Analysis-I

parameter c	power law $k \to 0$	Taylor's scale law	law for u'^2	power law $k \to \infty$
1/2	$f(x) \sim x$	$\lambda^2 = 10\nu t$	$u'^2 \sim t^{-1}$	$E(k,t) \sim x^{-\frac{5}{3}}$
2/5	$f(x) \sim x^2$	$\lambda^2 = \frac{25}{3}\nu t$	$u'^2 \sim t^{-\frac{6}{5}}$	$E(k,t) \sim x^{-\frac{5}{3}}$
1/3	$f(x) \sim x^3$	$\lambda^2 = \frac{15}{2}\nu t$	$u'^2 \sim t^{-\frac{4}{3}}$	$E(k,t) \sim x^{-\frac{5}{3}}$
.32	$f(x) \sim x^{3.25}$	$\lambda^2 = 7.35\nu t$	$u'^2 \sim t^{-1.36}$	$E(k,t) \sim x^{-\frac{5}{3}}$
2/7	$f(x) \sim x^4$	$\lambda^2 = 7\nu t$	$u'^2 \sim t^{-\frac{10}{7}}$	$E(k,t) \sim x^{-\frac{5}{3}}$
.285	$f(x) \sim x^{4.02}$	$\lambda^2 = 6.99\nu t$	$u'^2 \sim t^{-1.43}$	$E(k,t) \sim x^{-\frac{5}{3}}$

Table 7.2: Analysis-II

parameter c	value of α	value of $\frac{7}{15}G_0$
1/2	1	2
2/5	6/5=1.2	11/6=1.83
1/3	4/3=1.3	7/4=1.75
.32	1.36	17/10=1.735
2/7	10/7=1.428	17/10=1.70
.285	1.43	17/10=1.699

Heisenberg's turbulence energy spectrum $f(x)$ of one parameter family of functions, the parameter being Reynolds number R_λ is given by,

$$\int_0^\infty f(x)dx - \frac{1}{2}xf(x) = 2[\frac{1}{\alpha} + \int_x^\infty (\frac{f(x)}{x^5})^{\frac{\lambda}{2}}dx] \int_0^x x^2 f(x)dx \tag{7.7.1}$$

$$R_\lambda^2 = \frac{u'^2\lambda^2}{\nu^2} = \frac{2}{3}\frac{\lambda^2}{\nu^2}\int_0^\infty E(k)dk = \frac{2}{3}\frac{\lambda^2}{\nu\epsilon}\alpha k^{-2}I_0, \tag{7.7.2}$$

where $I_0 = \int_0^x f(x)dx$, $\lambda^2 = 10\gamma t$

For self-preserving $\lambda^2 R_\lambda^2 = \frac{20}{3}\alpha I_0$.

On Loitsiansky's assumption of the rapidly vanishing of the correlation function e.g., $\lim_{r\to\infty} r^4 f' = 0$ and $\lim_{r\to\infty} r^4\lambda = 0$, the following relation holds

$$u'^2 \int_0^\infty f(r)r^4 dr = J_0 \tag{7.7.3}$$

where J_0 is a constant. Batchelor(1949) also showed that

$$\lim_{k\to 0}\frac{E(k,\epsilon)}{k^4} = J_0 \tag{7.7.4}$$

This indicates that spectral energy function $E(k,t)$ behaves at small k, $E(k,t) \sim k^4$. Thus we may conclude that this point of the energy spectrum does not belong to the similar spectrum of Heisenberg e.g., $u'^2 \sim t^{-1}$, $\lambda^2 = 10\nu t$, $E(k,t) \sim k(k \to 0)$, $E(k,t) \sim k^{\frac{-5}{3}}(k \to \infty)$.

In a recent research article by Meldi and Segant([17]) discussed measures that "for arbitrary values of the decay exponent of Kinetic energy e.g.,$E(t) \sim t^{n_k}$ with $n_k \leq -1$ may be obtained arbitrarily neglecting some terms in the equation for prescribing some invariant quantities as found by $Von - K\acute{a}rm\acute{a}n - Howarth$, or an equivalent way according to Noether's theorem, enforcing a symmetry in the problem."

Also with the consideration of single velocity and a single length scale, George's theory([6]) lead to a general decay regime with $l(t) \sim t^{\frac{1}{2}}$ and $K(t) \sim t^{n_k}$, where n_k is fixed by the initial conditions or the existence of an invariant quantity. The classical case $n_k = -1$ can be recovered easily.

7.8 Concluding remarks

As a consequence, still keeping the same theoretical framework, any departure from this regime should be interpreted as a breakdown of exact self-preservation. Nevertheless, small departures from exact self-preservation are expected to yield $K(t) \sim t^{-1}$ as exact self-preservation does.

In the present analysis, we have been able to discuss that fixed-point method may be applied well to the turbulence problems when it is characterized by a large Reynolds number and the flow permits comparatively for a large time. It is shown that the energy spectrum $E(k,t)$ should exist as self preserving solution for the entire spectrum (Heisenberg's case). During the early period decay the nonlinear inertia term may dominate over viscous dissipation terms characterized by the parameter c having values between 0 and 2/3. That is, $K\acute{a}rm\acute{a}n - Howarth$ equation may correspond to the case when ν is negligible as in Sen's treatment. In the Heisenberg's case for $c = \frac{1}{2}$ the energy decay law is $u'^2 \sim t^{-1}$ while $c = \frac{2}{7}$

the decay law applies even up to $u'^2 \sim t^{-\frac{10}{7}}$ due to Kolmogorov or for $c = .285$ where $u'^2 \sim t^{-1.43}$ that mentioned by Chasnov([5]).The fact that ought to be mentioned is the Sen's solution may be applied to the large Reynolds number cases completely. While in the last case, for example the case $c = \frac{2}{7}$ is valid for the final period decay.To avoid discrepancy Lin(1947)([25]) prescribed that $u'^2 = \alpha t^{-1} + \beta$ where α, β are constant. Lin(1947)([25]) showed further that Heisenberg's case is very probably stable when viscosity term remain present in the energy decay equation while all the cases are unstable when viscous dissipation is negligible as shown by Mazumdar([15]). We may encompass the idea of George([6]) the energy spectrum approach may also be applied suitably to analyze the decay of energy spectrum in homogeneous isotopic turbulence. Finally, we may conclude that the present fixed point approach may be applied in a straight forward manner to all the cases discussed here above.

[0]Some of the materials presented in this chapter are taken from our paper (see-[11]) entitled "Fixed Point Analysis of a Homogeneous Isotropic Turbulent flow", (communicated to).

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