CHAPTER III

TRANSVERSE VIBRATION OF PLATES BASED ON ELASTIC FOUNDATION.
3.1. **Transverse Vibration of a Non-Isotropic Plate of Rectangular and Elliptic Boundaries Resting on Elastic Foundation.**

**INTRODUCTION**

The problem of transverse vibration of isotropic solids of different shapes with various boundary conditions was considered by several authors. The problem becomes complicated when the material is non-isotropic. Some authors discussed the same problem with orthotropic materials.

In the present paper the problem of transverse vibration is considered for a plate of certain interesting type of anisotropic material with (i) rectangular and (ii) elliptic boundaries resting on elastic foundation. In the former case an exact solution is obtained for plate simply-supported on all its boundaries. In the case of an elliptic boundary, the solution can be obtained by using Mathieu functions. But even with these functions certain approximations are required for deducing the frequencies. To find approximately the gravest mode of vibration it is believed that the simple method of Galerkin is adequate. Hence this method is used in this case.

**FORMULATION OF THE PROBLEM**

Let us take

\[ \omega \]  
- displacement normal to the middle plane which was initially in the \( xy \)-plane;

\[ \sigma_x, \sigma_y \]  
- stresses along \( x \) and \( y \) axes respectively;

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\( \varepsilon_x, \varepsilon_y \) — strain components along the \( x \) and \( y \) axes respectively;

\( E', E'', E'', G \) — elastic constants;

\( \tau_{xy} \) — shearing stress;

\( \gamma_{xy} \) — shearing strain in the \( xy \)-plane;

\( h \) — thickness of the plate;

\[
D_x = \frac{E_x h^3}{12}, \quad D_y = \frac{E_y h^3}{12}, \quad D_{xy} = \frac{G h^3}{12}, \quad D_i = \frac{E'' h^3}{12}
\]

\( k \) — modulus of elastic foundation;

\( \omega \) — angular frequency;

\( 2a, 2b \) — lengths of two perpendicular edges in the case of a rectangular plate and major and minor axes in the case of elliptic plate;

\( \rho \) — density of the material of the plate.

Let the plate be assumed to have three orthogonal planes of elastic symmetry. If they are considered as the principal planes and the co-ordinate planes are taken along them then the relations between the components of stress and strain in the \( xy \)-plane are represented by [21]

\[
\begin{align*}
\sigma_x &= E' \varepsilon_x + E'' \varepsilon_y \\
\sigma_y &= E' \varepsilon_y + E'' \varepsilon_x \\
\tau_{xy} &= G \gamma_{xy}
\end{align*}
\]

(3.1.1)

If the material of the plate be such that the elements normal to the middle plane remain normal to it after they are strained, then the strain components are given by

\[
\begin{align*}
\varepsilon_x &= - \chi \frac{\partial \omega}{\partial x^2} \\
\varepsilon_y &= - \chi \frac{\partial \omega}{\partial y^2}
\end{align*}
\]
\[ \gamma_{xy} = -2 \pi \frac{\partial^2 \omega}{\partial x \partial y} \quad (3.1.2) \]

The bending and twisting moments are

\[ M_x = \int_{-h/2}^{h/2} x \sigma_x \, dx \]
\[ = - \left( D_1 \frac{\partial^3 \omega}{\partial x^2} + D_3 \frac{\partial^2 \omega}{\partial y^2} \right) \]
\[ M_y = \int_{-h/2}^{h/2} x \sigma_y \, dx \]
\[ = - \left( D_2 \frac{\partial^2 \omega}{\partial y^2} + D_3 \frac{\partial^2 \omega}{\partial x^2} \right) \]
and
\[ M_{xy} = -\int_{-h/2}^{h/2} x \tau_{xy} \, dx = 2 D_{xy} \frac{\partial^3 \omega}{\partial x \partial y} \]

If the plate is placed on elastic foundation then the reaction of the subgrade at any point of the bottom plane is proportional to the normal displacement \( \omega \) and is given by \(-k \omega\), where \( k \) is the modulus of elastic foundation (it is measured in pounds per square inch of area for one inch of deflection). It is constant for the particular material used in the plate.

If \( q \) be the lateral load acting on the plate then the total transverse loading is \( q - k \omega \). In the case of vibration of the plate, the transverse load is given by

\[ q = -\rho \frac{\partial^2 \omega}{\partial t^2} \quad (3.1.10) \]

The equation of vibration of the plate is
Substituting the values of \( M_x, M_y, M_{xy} \) and \( q \) from the equations (3.1.3) and (3.1.4) the equation (3.1.5) takes the form:

\[
D_x \frac{\partial^4 w}{\partial x^4} + 2H \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 w}{\partial y^4} + \rho \frac{\partial^2 w}{\partial t^2} + kw = 0 ,
\]

(3.1.6)

where

\[
H = D_x + 2D_{xy} ,
\]

(3.1.7)

Then following Ruber's suggestion, \( H \) is assumed to be the geometric mean of \( D_x \) and \( D_y \) i.e.

\[
H = \sqrt{D_x D_y}
\]

This result is approximately true in the case of concrete.

With this value of \( H \), the differential equation of vibration of the plate becomes

\[
D_x \frac{\partial^4 w}{\partial x^4} + 2\sqrt{D_x D_y} \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 w}{\partial y^4} + \rho \frac{\partial^2 w}{\partial t^2} + kw = 0 ,
\]

(3.1.8)

**SOLUTION OF THE PROBLEM**

For normal type of vibration, \( w \) can be assumed as:

\[
w(x, y, t) = W(x, y) \cos(b_{mn} t + \epsilon)
\]

(3.1.9)

Where \( b_{mn} \) is the angular frequency and \( \epsilon \) is a constant.

Substituting this value of \( w \) in the equation (3.1.8) we get

\[
D_x \frac{\partial^4 W}{\partial x^4} + 2\sqrt{D_x D_y} \frac{\partial^4 W}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 W}{\partial y^4} + \lambda_{mn} W = 0 ,
\]

(3.1.10)

where

\[
\lambda_{mn} = k - \rho b_{mn}^2
\]

(3.1.11)
The equation (3.1.10) can be written as:

\[
(\sqrt{D_x} \frac{\partial^2}{\partial x^2} + \sqrt{D_y} \frac{\partial^2}{\partial y^2})(\sqrt{D_x} \frac{\partial^2}{\partial x^2} + \sqrt{D_y} \frac{\partial^2}{\partial y^2})W + \lambda_{mn} W = 0 ,
\]

Putting \[x = x, D_x^{\frac{1}{4}}, y = y, D_y^{\frac{1}{4}}\]

the equation (3.1.12) becomes

\[
\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)W + \lambda_{mn} W = 0 ,
\]

Case I. The plate is rectangular with \(2a\) and \(2b\) as its length and breadth taken along the axes of \(x\) and \(y\) respectively. If the plate is simply supported on all the boundaries, then

\[W = 0 = \frac{\partial^2 W}{\partial x^2}\] on \(x = 0\) and \(x = 2a\); and

\[W = 0 = \frac{\partial^2 W}{\partial y^2}\] on \(y = 0\) and \(y = 2b\),

with the origin at one corner of the plate.

Using the transformation (3.1.13), the above boundary conditions are transformed into:

\[W = 0\] at \(x_1 = y_1 = 0\) and \(x_1 = 2a_1, y_1 = 2b_1\),

\[\frac{\partial^2 W}{\partial x_1^2} = 0\] on \(x_1 = 2a_1\), and \[\frac{\partial^2 W}{\partial y_1^2} = 0\] on \(y_1 = 2b_1\)

where \(x_1 = \frac{2a}{\sqrt{D_x}} = a_1\), \(y_1 = \frac{2b}{\sqrt{D_y}} = b_1\)
To satisfy the above boundary conditions, \( W \) is assumed in the form
(Navier \textsuperscript{[24]} solution)

\[
W = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m \pi x}{a_1} \sin \frac{n \pi y}{b_1},
\]

where \( A_{mn} \) is an arbitrary constant and \( m, n \) are any two positive integers.

Substituting the value of \( W \) in the equation (3.1.14) we get the frequency equation in the form:

\[
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[ \left( \frac{m^2}{a_1^2} + \frac{n^2}{b_1^2} \right)^2 n^4 + \lambda_{mn} \right] = 0
\]

Therefore, by (3.1.11) we get

\[
\beta_{mn}^2 = \frac{1}{\rho} \left[ \left( \frac{m^2}{a_1^2} + \frac{n^2}{b_1^2} \right)^2 n^4 + \kappa \right]
\]

The gravest mode of vibration corresponds to \( m = n = 1 \) and is given by

\[
\beta_{11}^2 = \frac{1}{\rho} \left[ \left( \frac{\sqrt{a_1^2} + \sqrt{b_1^2}}{a_1^2} \right)^2 \cdot \frac{n^4}{16} + \kappa \right].
\]

**Case II**  \hspace{1cm} The plate is elliptic having the equation

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1
\]

This boundary can be transformed into another ellipse by the substitution (3.1.15) and we get
\[
\frac{x_i^2}{a_i^2} + \frac{y_i^2}{b_i^2} = 1,
\]

where

\[
a_i = \frac{a}{\sqrt{D_x}}, \quad b_i = \frac{b}{\sqrt{D_y}},
\]

To get an approximate solution of this problem, Galerkin's method is used with the substitution

\[
W = a_o \left[ 1 - \frac{x_i^2}{a_i^2} - \frac{y_i^2}{b_i^2} \right],
\]

involving only one parameter \( a_o \). This value of \( W \) satisfies the boundary conditions, for a clamped base:

\[
W = \frac{\partial W}{\partial x_i} = \frac{\partial W}{\partial y_i} = 0
\]

on the elliptic boundary.

Substituting the value of \( W \) in the left hand member of the equation (3.1.14) an error function \( \varepsilon_i(x_i, y_i) \) is obtained and is given by

\[
\varepsilon_i(x_i, y_i) = \frac{4a_o}{(a_i b_i)^4} \left[ 3(a_i^4 + b_i^4) + 2a_i^2 b_i^2 \right] + \lambda a_o \left( -\frac{x_i^2}{a_i^2} - \frac{y_i^2}{b_i^2} \right)^2
\]

This error is then minimised by the orthogonality conditions:

\[
\iint_{R} \varepsilon_i(x_i, y_i) W \, dx_i \, dy_i = 0
\]

integrated over the entire area \( R \) of the elliptic plate. This gives
the value of \( p^2 \), which corresponds to the gravest mode of vibration. The value of \( p^2 \) is given by

\[
p^2 = \frac{4 \left[ 3 \left( \frac{1}{a^4} + \frac{1}{b^4} \right) + \frac{2}{(a^2 b^2)^3} + \frac{\kappa^{539}}{128} \right]}{\rho \frac{539}{128}}
\]

\[
= \frac{1}{\rho} \left[ \kappa + 0.95 \left\{ 3 \left( \frac{Dx}{a^4} + \frac{Dy}{b^4} \right) + 2 \sqrt{\frac{Dx \cdot Dy}{(ab)^4}} \right\} \right]
\]
3.2. TRANSVERSE VIBRATION OF AN ISOTROPIC CIRCULAR PLATE RESTING ON PASTERNAK-TYPE FOUNDATION

INTRODUCTION

Considering the safety factor of space-crafts and industrial installations the machineries involved such as solid rocket propellant motors etc., the use of soft filaments now-a-days seems to be very popular. These filaments are mainly elastic and sometimes visco-elastic foundations. Some models of them have been recently discussed by Kerr\textsuperscript{25} (1964).

In this paper, we have considered the problem of transverse vibration of an isotropic circular plate of moderate thickness and resting on Pasternak-type foundation. An exact solution is obtained for the circular plate clamped at the rim.

FORMULATION OF THE PROBLEM

Let us take
\[ \omega = \text{normal displacement of any point of the plate}; \]
\[ h = \text{thickness of the plate}; \]
\[ \rho = \text{density of the material of the plate}; \]
\[ D = \frac{Eh^3}{12(1-\sigma^2)} = \text{flexural rigidity}; \]
\[ E = \text{Young's modulus}; \]
\[ \sigma = \text{Poisson's ratio}; \]
\[ \rho = \text{interface pressure}; \]

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The differential equation of motion of the plate is

\[ D \nabla^4 \omega + \rho h \frac{\partial^2 \omega}{\partial t^2} + p = 0 \]  \hspace{1cm} (3.2.1)

For a foundation model of Pasternak-type (Fig.6), \( p \) is given by

\[ p = k \omega - G \nabla^2 \omega \]  \hspace{1cm} (3.2.2)

where

\( k \) = foundation constant;

and \( G \) = shear modulus of the plate.

Thus the equation (3.2.1) becomes

\[ \nabla^4 \omega + \frac{\rho h}{D} \frac{\partial^2 \omega}{\partial t^2} + \frac{k \omega}{D} - \frac{G}{D} \nabla^2 \omega = 0 \]  \hspace{1cm} (3.2.3)

For a normal type of vibration, \( \omega \) can be assumed as:

\[ \omega = We^{i \omega t} \]

where \( W \) is independent of \( t \). Then the equation (3.2.3) becomes

\[ \nabla^4 W - \alpha_1 \nabla^2 W + \alpha_2 W = 0 \]  \hspace{1cm} (3.2.4)

where

\[ \alpha_1 = \frac{G}{D} \hspace{1cm} \alpha_2 = \frac{k - \rho h \omega^2}{D} \]  \hspace{1cm} (3.2.5)

The equation (3.2.4) can be written as

\[ (\nabla^2 - \alpha)(\nabla^2 - \beta)W = 0 \]  \hspace{1cm} (3.2.6)
(Pasernak Foundation model)

Fig. 6
\( \alpha, \beta \) being the roots of the equation
\[
x^2 - \alpha_1 x + \alpha_2 = 0,
\]
so that
\[
\alpha + \beta = \alpha_1 = \frac{G}{D}, \quad \alpha \beta = \alpha_2 = \frac{K - \rho h \omega^2}{D},
\]
Let
\[
\alpha = \frac{G + \sqrt{G^2 - 4D(K - \rho h \omega^2)}}{2D}, \quad \beta = \frac{G - \sqrt{G^2 - 4D(K - \rho h \omega^2)}}{2D},
\]
The roots are real if
\[
\omega^2 \geq \frac{k}{\rho h} - \frac{G^2}{4 \rho h D},
\]
and then the general solution of the equation (3.2.6) is
\[
W = A_1 I_\alpha (\pi \sqrt{\alpha}) + B_1 K_\alpha (\pi \sqrt{\beta}) + C_1 I_\alpha (\pi \sqrt{\beta}) + D_1 K_\alpha (\pi \sqrt{\alpha}),
\]
where \( A_1, B_1, C_1, D_1 \) being arbitrary constants and \( I_\alpha, K_\alpha \) being modified Bessel functions of order zero.

For finite values of \( W \) at the centre ( \( \rho = 0 \) ), \( B_1, D_1 \) must vanish identically, so that
\[
W = A_1 I_\alpha (\pi \sqrt{\alpha}) + C_1 I_\alpha (\pi \sqrt{\beta}),
\]

**BOUNDARY CONDITIONS**

The plate is assumed to be clamped at the rim \( \rho = a \), \( a \) being the radius of the plate. Therefore,
\[ W = \frac{dW}{dn} = 0 \quad \text{at} \quad n = a \]

Thus applying these boundary conditions, the equation (3.2.10) gives

\[ A_1 I_0(a\sqrt{\alpha}) + C_1 I_0(a\sqrt{\beta}) = 0 \]

and

\[ \sqrt{\alpha} A_1 I_0'(a\sqrt{\alpha}) + \sqrt{\beta} C_1 I_0'(a\sqrt{\beta}) = 0 \]

Eliminating \( A_1 \) and \( C_1 \), from these equations we get the frequency equation in the form:

\[ \frac{\sqrt{\alpha} I_0'(a\sqrt{\alpha})}{I_0(a\sqrt{\alpha})} = \frac{\sqrt{\beta} I_0'(a\sqrt{\beta})}{I_0(a\sqrt{\beta})}, \quad (3.2.11) \]

Let us put

\[ a\sqrt{\alpha} = \nu, \quad a\sqrt{\beta} = \psi \quad \text{and} \quad \nu = \lambda \psi, \quad (3.2.11) \]

Then the equation (3.2.11) can be written as

\[ \nu \lambda I_0'(\lambda \psi) I_0(\psi) - \nu I_0'(\nu) I_0(\lambda \psi) = 0. \]

Now \( I_o'(x) = I_1(x) \) and \( \nu = a\sqrt{\beta} \neq 0 \), so that we get

\[ \lambda I_1(\lambda \psi) I_0(\nu) - I_1(\psi) I_0(\lambda \psi) = 0. \quad (3.2.12) \]

This equation can be written in the expanded form:

\[ (\lambda^2 - 1) \left[ 1 + \left( \frac{\psi}{2} \right)^2 \frac{\lambda^2 + 1}{2} + \left( \frac{\psi}{2} \right)^4 \frac{\lambda^4 + 4\lambda^2 + 1}{12} + \left( \frac{\psi}{2} \right)^6 \frac{(\lambda^2 + 1)(\lambda^4 + 8\lambda^2 + 1)}{144} + \ldots \right] = 0. \]

The expression within the square bracket is a sum of positive quantities for all real values of \( \psi \) and \( \lambda \). Hence this expression cannot be equal to zero. Thus the above equation gives the only real
root

\[ \lambda^2 = 1 \]

which corresponds to \( \alpha = \beta \). In that case, the roots of the equation (3.2.7) are real and equal and therefore,

\[ \omega^2 = \frac{4 DR - G^2}{4 \rho h D} \quad (3.2.14) \]

clearly, \( \omega \) will be real, if

\[ G^2 \leq 4 DR \]

\( G^2 = 4 DR \) corresponds to the static case, and if

\( G^2 < 4 DR \) then the only real value of \( \omega^2 \) is given by the relation (3.2.14)