6.1. TRANSVERSE VIBRATION OF A COMPOSITE CIRCULAR MEMBRANE

INTRODUCTION

The problem of transverse vibration of a composite circular membrane was discussed by Volicka [5] (1962). In that paper the circular membrane was piece-wise homogeneous.

In the present paper the membrane considered consists of two concentric circular regions. The inner region being composed of homogeneous material and the density of the outer region varying inversely as the square of the distance from the centre.

FORMULATION OF THE PROBLEM

Let

- \( \circ \) - centre of the circular membrane (fig. 8)
- \( a \) - radius of the inner circle;
- \( b \) - radius of the outer circle;
- \( \rho_0 \) - density of the material of the membrane in \( 0 \leq \kappa \leq a \);
- \( \frac{\rho_0 a^2}{\pi^2} \) - density of the material of the membrane in \( a \leq \kappa \leq b \);
- \( \omega \) - normal displacement of any point \( P(\kappa, \theta) \) of the membrane from the position of stable equilibrium;
- \( T \) - uniform tension of the membrane.

The equation of motion of the membrane is

\[
\frac{T}{\rho} \left( \frac{\partial^2 \omega}{\partial \kappa^2} + \frac{1}{\kappa} \frac{\partial \omega}{\partial \kappa} + \frac{1}{\pi^2} \frac{\partial^2 \omega}{\partial \theta^2} \right) = \frac{\partial^2 \omega}{\partial t^2}
\]

* Published in Revue de Mechaniques Appliquees, Vol. 12, No. 3 (1967) pp. 607-609.
For symmetric displacement about the origin, \( \omega \) is independent of \( \theta \), so that
\[
\frac{T}{\rho} \left( \frac{\partial^2 \omega}{\partial n^2} + \frac{i}{n} \frac{\partial \omega}{\partial n} \right) = \frac{\partial^2 \omega}{\partial t^2},
\]
(6.1.1)

For harmonic type of motion, we can write
\[
\omega = u \cos \beta t, \quad (6.1.2)
\]
where \( u \) is independent of \( t \).

From the equations (6.1.1) and (6.1.2) we have
\[
\frac{d^2 u}{dn^2} + \frac{i}{n} \frac{du}{dn} + \frac{\rho \beta^4 u}{T} = 0, \quad (6.1.3)
\]

In the region \( 0 \leq n \leq a \), \( \rho = \rho_0 \). Hence the equation (6.1.3) becomes
\[
\frac{d^2 u}{dn^2} + \frac{i}{n} \frac{du}{dn} + \alpha^2 u = 0, \quad (6.1.4)
\]
where
\[
\alpha^2 = \frac{\beta^2 \rho_0}{T}, \quad (6.1.5)
\]

The general solution of the equation (6.1.4) is
\[
u = A J_0(\alpha n) + B Y_0(\alpha n),
\]
\(A, B\) being two arbitrary constants.

For finite values of \( u \) at \( n = 0 \), we have \( B = 0 \) and therefore
\[
u = A J_0(\alpha n), \quad (6.1.6)
\]

Again, in the region \( a \leq n \leq b \), \( \rho = \frac{\rho_0 \alpha^2}{n^2} \). Hence, the equation (6.1.3) takes the homogeneous form:
\[ r^2 \frac{d^2 u}{dr^2} + r \frac{du}{dr} + (\alpha^2) u = 0, \quad (6.1.7) \]

Putting \( r = e^y \) in the above equation, we get

\[ \frac{d^2 u}{d y^2} + (\alpha^2) u = 0. \]

The general solution of this equation is

\[ u = A \cos (\alpha y) + B \sin (\alpha y) \]

\[ = A \cos (\alpha \log_e r) + B \sin (\alpha \log_e r), \quad (6.1.8) \]

The membrane is assumed to be clamped at the outer rim, so that

\[ u = 0 \quad \text{on} \quad r = b \quad \text{and} \]

\[ 0 = A \cos (\alpha \log_e b) + B \sin (\alpha \log_e b) \]

Then from (6.1.8) we can write \( u \) in the form:

\[ u = K \left[ \sin(\alpha \log_e r - \alpha \log_e b) \right] \]

\[ = K \sin \left[ \alpha \log_e \left( \frac{r}{b} \right) \right], \quad (6.1.9) \]

Where \( K \) is some arbitrary constant.

From (6.1.6) and (6.1.9) we get the displacement of any point in the inner and outer regions respectively.

On the common boundary ( \( r = a \) ) \( u \) and \( \frac{du}{dr} \) will be equal from both sides, so that

\[ A J_0(a \alpha) = K \sin \left[ \alpha \log_e \left( \frac{a}{b} \right) \right] \]

and

\[ A J'_0(a \alpha) = K \alpha \cos \left[ \alpha \log_e \left( \frac{a}{b} \right) \right] \]

Using the relation

\[ J'_0(a \alpha) = -\alpha J_1(a \alpha) \]
we get the frequency equation:

\[ J_0 \left( \alpha \delta \right) \cos \left[ \alpha \delta \log_e \left( \frac{a}{b} \right) \right] + J_1 \left( \alpha \delta \right) \sin \left[ \alpha \delta \log_e \left( \frac{a}{b} \right) \right] = 0 \]

In particular, if we put \( \alpha = 5 \text{ cms.} \), \( b = 10 \text{ cms.} \), then

\[ J_0 \left( 5 \delta \right) \cos \left( 5 \delta \log_e 2 \right) - J_1 \left( 5 \delta \right) \sin \left( 5 \delta \log_e 2 \right) = 0 \]

From this equation it is found that for small frequency i.e. for small values of \( \alpha \),

\[ p^2 = 0.056556 \cdot \frac{r}{p^0} \]
6.2. NOTE ON THE TRANSVERSE VIBRATION OF A CIRCULAR MEMBRANE OF VARIABLE DENSITY *

INTRODUCTION

In a recent paper, Sen Gupta and Ghosh [6] have solved the problem of transverse vibration of a circular membrane with the density varying as $\exp(-\varepsilon \pi^2)$ where $\pi$ is the distance from the centre of the membrane and $\varepsilon$ is a constant.

In the present paper, a similar problem of the membrane having the density varying as $\exp(-\epsilon \pi)$ has been solved by the perturbation method.

Nomenclature

- $T$ - Uniform tension of the membrane;
- $\rho$, $\theta$ - polar coordinates of any point $P$ on the membrane;
- $\omega$ - small normal displacement of $P$ from the equilibrium position of the membrane in time $t$ seconds;
- $a$ - radius of the membrane;
- $\rho$ - density of the material of the membrane;
- $\rho_0$ - value of $\rho$ at $r = 0$;
- $u$ - normal coordinate independent of $t$;
- $\phi, \chi, \mu, \nu$ - perturbation elements;

$$\lambda = \frac{\rho_0^2}{T}$$ where $\frac{2\pi}{\rho}$ is the period of vibration;

$\omega$ is a function of $r, \theta, t$; $\mu$ is the displacement factor

* Published in Indian Jr. of Theoretical Physics, Vol. 14, No. 3, Sept. 1966, pp. 72-79.
independent of $t$; $\psi$ is the displacement factor independent of $a$

and $t$.

$L_n$ is an operator, given by $L_n \equiv \frac{d}{dr} (r \frac{d}{dr}) - \frac{j^2}{r}$, $j$ being

integers.

$\bar{\Psi}_{jn}, \bar{\lambda}_{jn}$ perturbed eigenfunctions and eigen-values respectively.

The equation of motion $^{[33]}$ is

$$\frac{T}{\rho} \left[ \frac{\partial^2 \omega}{\partial r^2} + \frac{1}{r} \frac{\partial \omega}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \omega}{\partial \theta^2} \right] = \frac{\partial^2 \omega}{\partial t^2}, \quad (6.2')$$

Let

$$\omega = u \cos (bt + \alpha)$$

where $u$ is independent of $t$ and $\alpha$ is an arbitrary constant.

Then the equation (6.2.1) becomes

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\rho b^2 u}{T} = 0, \quad (6.2').$$

If

$$u = \sum_j \psi_j(r) \cos (\theta + \alpha_j),$$

where $\alpha_j$ is a constant angle, the equation (6.2.2) can be written as

$$\frac{d^2 \psi_j}{dr^2} + \frac{1}{r} \frac{d \psi_j}{dr} + \left( \frac{\rho b^2}{T} - \frac{j^2}{r^2} \right) \psi_j = 0.$$

Let

$$\rho = \rho_0 e^{-\epsilon r},$$

So that

$$\frac{d^2 \psi_j}{dr^2} + \frac{1}{r} \frac{d \psi_j}{dr} + \left( \lambda e^{\epsilon r} - \frac{j^2}{r^2} \right) \psi_j = 0.$$
where
\[ \lambda = \frac{p^2/\alpha}{r} \]

\[ \therefore \frac{d^2\Psi_j}{d\pi^2} + \frac{i}{\pi} \frac{d\Psi_j}{d\pi} + \left( \lambda - \frac{j^2}{\pi^2} \right)\Psi_j = \left[ \epsilon(\lambda \pi) - \frac{\epsilon^2}{3} (\lambda \pi^2) + \ldots \right] \Psi_j, \quad (6.2.1) \]

If \( \epsilon = 0 \) it becomes

\[ \frac{d^2\Psi_{jn}}{d\pi^2} + \frac{i}{\pi} \frac{d\Psi_{jn}}{d\pi} + \left( \lambda - \frac{j^2}{\pi^2} \right)\Psi_{jn} = 0, \quad (6.2) \]

\( \Psi_{jn} \) being the \( n \) th eigen-function of the above equation.

Then using the operator

\[ L_n \equiv \frac{d}{d\pi} \left( r \cdot \frac{d}{d\pi} \right) - \frac{j^2}{\pi} \]

the equation (6.2.4) can be written as

\[ L_n \left( \Psi_{jn} \right) + \lambda_{jn} r \Psi_{jn} = 0 \quad \ldots \quad (6.2) \]

If \( \epsilon \) is so small that \( \epsilon^3, \epsilon^4, \ldots \), can be neglected, then the equation (6.2.3) becomes

\[ L_n \left( \Psi_{jn} \right) + r \bar{\lambda}_{jn} \Psi_{jn} = \left[ \epsilon \bar{\lambda}_{jn} r^2 - \frac{\epsilon^2}{2} r^3 \bar{\lambda}_{jn} \right] \Psi_{jn}, \quad (6.2) \]

where \( \bar{\Psi}_{jn} \) - perturbed eigen functions,
\( \bar{\lambda}_{jn} \) - perturbed eigen values:

writing

\[ \Psi_{jn} = \Psi_{jn} + \epsilon \phi_{jn} + \epsilon^2 \chi_{jn} \]
\[ \bar{\Psi}_{jn} = \bar{\lambda}_{jn} + \epsilon \mu_{jn} + \epsilon^2 \nu_{jn} \quad \ldots \quad (6.3) \]
where $\phi_{jn}, \lambda_{jn}$ are the perturbation elements of $\bar{\psi}_{jn}$ and 
$\mu_{jn}, \nu_{jn}$ are the perturbation elements of $\bar{\lambda}_{jn}$.

In the equations (6.2.6) and (6.2.7) and by equating co-efficients
of different powers of $\varepsilon$, we have

$$L_n (\psi_{jn}) + \pi \lambda_{jn} \psi_{jn} = 0, \quad (6.2.8)$$

$$L_n (\phi_{jn}) + \pi \lambda_{jn} \phi_{jn} = \pi^2 \lambda_{jn} \psi_{jn} - \pi \mu_{jn} \psi_{jn}, \quad (6.2.9)$$

and

$$L_n (\lambda_{jn}) + \pi \lambda_{jn} \lambda_{jn} = \lambda_{jn} (\pi^2 \phi_{jn} - \frac{\pi^2}{2} \psi_{jn}) - \mu_{jn} (\pi \phi_{jn} - \pi^2 \psi_{jn}) - \pi \psi_{jn} \nu_{jn}, \quad (6.2)$$

Clearly, equations (6.2.8) and (6.2.5) are identical and they

correspond to the unperturbed state of vibration.

For the unperturbed state of vibration, the solution is

$$\psi_{jn} (x) = J_j (ax \sqrt{\lambda_{jn}}), \quad (6.2.1)$$

where

$$x = \pi / a \quad (6.2.1)$$

On the fixed boundary, $u = 0$, $r = a$ so that

$$\psi_{jn} (x) = 0 \quad \text{at} \quad x = 1$$

and

$$J_j (a \sqrt{\lambda_{jn}}) = 0 \quad (6.2.1)$$

Again, equations (6.2.9) and (6.2.10) respectively can be written as

$$L_n (\phi_{jn}) + a^2 x \lambda_{jn} \phi_{jn} = a^2 x \lambda_{jn} \psi_{jn} - a^2 x \mu_{jn} \psi_{jn}, \quad (6.2)$$

and

$$L_n (\lambda_{jn}) + a^2 x \lambda_{jn} \lambda_{jn} = \lambda_{jn} (a^2 x \phi_{jn} - \frac{a^2}{2} x^3 \psi_{jn}) - a^2 x \psi_{jn} \nu_{jn}$$

$$- \mu_{jn} (a^2 x \phi_{jn} - a^3 x^2 \psi_{jn}), \quad (6.2)$$
1. First Order Calculations

The equation (6.2.14) is multiplied by \( \psi_{jm} \) and integrated with respect to \( z \) from \( z = 0 \) to \( z = 1 \) (i.e., over the entire membrane) so that

\[
\int_0^1 \psi_{jm} L_z (\phi_{jn}) \, dz + a^2 \lambda_{jn} \int_0^1 z \psi_{jm} \phi_{jn} \, dz = a^3 \lambda_{jn} \int_0^1 z^2 \psi_{jm} \psi_{jn} \, dz - a^2 \mu_{jn} \int_0^1 z \psi_{jm} \psi_{jn} \, dz. \tag{6.2.11}
\]

Writing

\[
\phi_{jn} = \sum_{k=1}^n A_{nk} \psi_{kn}
\]

and using (6.2.13) we get

\[
\int_0^1 z \psi_{jm} \phi_{jn} \, dz = \frac{1}{2} A_{nm} J_d^{\prime 2} (a \sqrt{\lambda_{jn}})
\]

Again by Green's Theorem,

\[
\int_0^1 \left[ \psi_{jm} L_z (\phi_{jn}) - \phi_{jn} L_z (\psi_{jm}) \right] \, dz = 0
\]

And by using the relations (6.2.5) and (6.2.13)

\[
\int_0^1 \psi_{jm} L_z (\phi_{jn}) \, dz = -\frac{1}{2} a^2 \lambda_{jn} A_{nm} J_d^{\prime 2} (a \sqrt{\lambda_{jn}})
\]

From equation (6.2.16) it can be obtained,

\[
\frac{1}{2} a^2 (\lambda_{jn} - \lambda_{jm}) J_d^{\prime 2} (a \sqrt{\lambda_{jn}}) A_{nm} = a^3 \lambda_{jn} \int_0^1 z^2 \psi_{jm} \psi_{jn} \, dz - \frac{1}{2} a^2 \mu_{jn} J_d^{\prime 2} (a \sqrt{\lambda_{jn}}) \delta_{mn}
\]

where \( \delta_{mn} = 1 \) for \( m = n \)

\( = 0 \) for \( m \neq n \).
When \( m \neq n \),

\[
A_{nm} = \frac{2a \lambda_j^i n}{(\lambda_j^i - \lambda_j^m) J_j^i (a \sqrt{\lambda_j^m})} \int_0^1 z^2 \psi_j^m \psi_j^n \, dz, \tag{68}
\]

And when \( m = n \),

\[
\mu_{jn} = \frac{2a \lambda_j^i n \int_0^1 z^2 J_j^i (az \sqrt{\lambda_j^m}) \, dz}{J_j^i (a \sqrt{\lambda_j^m})}, \tag{69}
\]

From the condition of normalisation,

\[
\int_0^1 z \psi_j^m \phi_j^n \, dz = 0
\]

\[
\frac{1}{2} A_{nn} J_j^i (a \sqrt{\lambda_j^m}) = 0
\]

which gives

\[
A_{nn} = 0 \tag{68}
\]

\[
\phi_j^n = 2a \lambda_j^i n \sum_{m=1}^{\infty} \frac{J_j^i (az \sqrt{\lambda_j^m}) \int_0^1 z^2 J_j^i (az \sqrt{\lambda_j^m}) J_j^i (az \sqrt{\lambda_j^m}) \, dz}{(\lambda_j^i - \lambda_j^m) J_j^i (a \sqrt{\lambda_j^m})}
\]

\[
= 0 \quad \text{when} \quad m = n
\]

II. Second Order Calculations

Multiplying the equation (6.2.15) by \( \psi_j^m \) and integrating with respect to \( z \) from \( z = 0 \) to \( z = 1 \), it can be obtained:

\[
\int_0^1 \psi_j^m L_z (\chi_j^m) \, dz + a^2 \int_0^1 z \psi_j^m \lambda_j^m \chi_j^m \, dz =
\]
\[ = \lambda_{jn} a^3 \int_0^1 z^2 \psi_j \phi_{jn} \, dz - \frac{1}{2} \lambda_{jn} a^4 \int_0^1 \psi_j \psi_{jn} \, dz - a^3 \mu_{jn} \int_0^1 z \phi_{jn} \psi_{jn} \, dz \]
\[ + a^3 \mu_{jn} \int_0^1 z^2 \psi_j \psi_{jn} \, dz - a^2 \nu_{jn} \int_0^1 z \psi_{jn} \psi_{jm} \, dz . \]

Assuming
\[ \chi_{jn} = \sum_{k=1}^{\infty} B_{nk} \psi_{jk} \]
and

proceeding as before, we get
\[ \frac{1}{2} a^2 B_{nm} (\lambda_{jn} - \lambda_{jm}) J_i^2 (a\sqrt{\lambda_{jn}}) = \lambda_{jn} a^3 \int_0^1 z^2 \psi_j \phi_{jn} \, dz - \frac{1}{2} a^4 \lambda_{jn} \int_0^1 \psi_j \psi_{jn} \, z^2 \, dz \]
\[ - \frac{1}{2} a^3 \mu_{jn} A_{nm} J_i^2 (a\sqrt{\lambda_{jn}}) + a^3 \mu_{jn} \int_0^1 z^2 \psi_j \psi_{jn} \, dz \]
\[ - a^2 \nu_{jn} \cdot \frac{1}{2} J_i^2 (a\sqrt{\lambda_{jn}}) \delta_{mn} . \]

where \( \delta_{mn} = 0 \) when \( m \neq n \) and \( \delta_{mn} = 1 \) when \( m = n \)

\[ \therefore \text{when } m = n, \text{ we have by using (6.2.18) and (6.2.19)} \]

\[ \nu_{jn} = \frac{2a \lambda_{jn} \int_0^1 z^2 \psi_j \phi_{jn} \, dz - a^2 \lambda_{jn} \int_0^1 z^2 \psi_{jn} \, dz}{J_i^2 (a\sqrt{\lambda_{jn}})} + \frac{(\mu_{jn})^2}{\lambda_{jn}} \]
and when \( m \neq n \)

\[ B_{nm} = \frac{a \lambda_{jn} \left[ 2 \int_0^1 z^2 \psi_j \phi_{jn} \, dz - a \int_0^1 z^2 \psi_j \psi_{jn} \, dz \right]}{(\lambda_{jn} - \lambda_{jm}) J_i^2 (a\sqrt{\lambda_{jn}})} - \frac{A_{nm} \lambda_{jm} \mu_{jn}}{\lambda_{jn} (\lambda_{jn} - \lambda_{jm})} \]

Also from the condition of normalisation
\[ \int_0^1 z \left[ \phi_{jn}^2 + 2 \psi_j \psi_{jn} \right] \, dz = 0 \]
From which it follows

\[ B_{nm} = - \frac{\int_0^1 z \phi_{jn}^2 \, dz}{J_j^2 (a\sqrt{\lambda_j})} \]

\[ \chi_{jn} = a\lambda_j n \sum_{m=1, m \neq n}^{\infty} \frac{\psi_{jm}}{\lambda_j n - \lambda_j m} \left[ \frac{2\int_0^1 z^2 \psi_{jm}^2 \phi_{jn} \, dz}{J_j^2 (a\sqrt{\lambda_j})} - \frac{A_{nm} \mu_{jm} \lambda_j m}{a(\lambda_j n)^2} ight] - \frac{\psi_{jn} \int_0^1 z \phi_{jn}^2 \, dz}{J_j^2 (a\sqrt{\lambda_j})} \]

With \( a\sqrt{\lambda_{on}} \) as the roots of the equation \( J_0 (a\sqrt{\lambda_{on}}) = 0 \), the following numerical results are obtained:

<table>
<thead>
<tr>
<th>( n )</th>
<th>( a\sqrt{\lambda_{on}} )</th>
<th>( a\mu_{on} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.405</td>
<td>.4304</td>
</tr>
<tr>
<td>2</td>
<td>5.520</td>
<td>.493</td>
</tr>
<tr>
<td>3</td>
<td>8.654</td>
<td>.493</td>
</tr>
</tbody>
</table>

It appears that as the eigenvalues increase the perturbation elements do not change much.