Chapter 1

Introduction

Let $G = (V, E)$ be a finite, simple, undirected graph. The vertex set of $G$ is denoted by $V(G)$ (or simply $V$) and edge set by $E(G)$ (or $E$). Each edge $e \in E$ is an unordered pair of distinct vertices of $V$. If an edge $e = uv$ then we say that a vertex $u$ is adjacent to vertex $v$ or the vertices $u$ and $v$ are neighbors and that $e$ is incident to $u$ and $v$. We use the notations $n = |V|$ and $m = |E|$ to denote the order and the size of $G$, respectively.

A subgraph $H$ of $G$ is a graph such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. An induced subgraph $H$ of $G$ (denoted by $< H >$) is a subgraph with the added property that if $u, v \in V(H)$, then $uv \in E(H)$ if and only if $uv \in E(G)$. For a vertex $v \in V(G)$, $G - \{v\}$ is called a vertex-deleted subgraph of $G$ and we write $G - v$ instead of $G - \{v\}$. If $F$ is a subset of edges of $G$ then a spanning subgraph of the graph $G$ is a subgraph containing all the vertices of $G$ and the edges $F \subseteq E(G)$. In particular, if $vu \in E(G)$, then the spanning subgraph $G - \{vu\}$ is called an edge-deleted subgraph of $G$. We write $G - vu$ instead of $G - \{vu\}$. If $u$ and $v$ are non-adjacent vertices of $G$, then $G + vu$ denotes the graph with vertex set $V(G)$ and edge set $E(G) \cup \{vu\}$. We follow Haynes et al. [10] for graph theoretical notations.

The degree of a vertex $v$ (denoted by $\deg(v)$) is equal to the number of vertices that are adjacent to $v$. The minimum degree (respectively, the maximum degree) of the graph $G$ is denoted by $\delta(G)$ (respectively, $\Delta(G)$ ). If there is a vertex $v \in V(G)$ such that $\deg(v) = 0$ then $v$ is called an isolated vertex. If $\deg(v) = 1$ then $v$ is called an end
vertex or a pendant vertex. A graph is said to be \( k \)-regular if each vertex has degree \( k \). For an edge set \( F \subseteq E \), \( V(F) \) is the set of end vertices of edges of \( F \).

A path is an alternating sequence of vertices and edges such that every edge joins the vertex preceding it with the vertex succeeding it and no vertex is repeated. If there is a path from every vertex in a graph \( G \) to every other vertex in \( G \) then \( G \) is said to be connected, otherwise \( G \) is disconnected.

The open neighborhood of a vertex \( v \) (denoted by \( N(v) \)) is the set of vertices of \( G \) that are adjacent to \( v \), that is \( N(v) = \{ u \in V \mid uv \in E \} \). The closed neighborhood of a vertex \( v \) is \( N[v] = N(v) \cup \{ v \} \). Let \( S \subseteq V \) and a vertex \( v \in S \) then the private neighborhood of \( v \) with respect to \( S \) (denoted by \( pn[v, S] \)) is the set of vertices \( \{ w \in V \mid N[w] \cap S = \{ v \} \} \). Also the external private neighborhood of \( v \) with respect to \( S \) (denoted by \( epn(v, S) \)) is the set of vertices \( \{ w \in V - S \mid N(w) \cap S = \{ v \} \} \) [6]. The external private neighborhood \( epn(v, S) \) of \( v \) with respect to \( S \) consists those private neighbors of \( v \) which are in \( V - S \). Thus, \( epn(v, S) = pn[v, S] \cap (V - S) \).

A dominating set \( S \subseteq V \) of \( G \) is a set of vertices such that each vertex \( v \in V \) is either in \( S \) or adjacent to a vertex of \( S \). The domination number (denoted by \( \gamma(G) \)) of \( G \) is the minimum cardinality of a dominating set of \( G \) [10]. Also \( S \) is a total dominating set if each vertex \( v \in V \) is adjacent to a vertex of \( S \). The total domination number (denoted by \( \gamma_t(G) \)) of \( G \) is the minimum cardinality of a total dominating set of \( G \) [10].

A vertex covering set \( S \subseteq V \) of \( G \) is a set of vertices such that every edge of \( G \) has at least one end vertex in \( S \). The vertex covering number (denoted by \( \alpha_0(G) \)) of \( G \) is the minimum cardinality of a vertex covering set of \( G \) [10].

An independent set \( S \subseteq V \) of \( G \) is a set of vertices such that no two vertices of \( S \) are adjacent. The independence number (denoted by \( \beta_0(G) \)) of \( G \) is the maximum cardinality of an independent set of \( G \) [10]. Note that \( \gamma(G) \leq \beta_0(G) \) for all graphs \( G \) [10].

In 1959, Gallai [7] presented his classical theorem involving the vertex covering number \( \alpha_0(G) \) and the vertex independence number \( \beta_0(G) \) as \( \alpha_0(G) + \beta_0(G) = n \), for any graph \( G = (V, E) \) with \( n \) vertices.

The domination, covering and independence in graph are few areas of research.
Some research papers have been published in these areas [3, 6, 15, 19]. We have con-
sidered the edge domination and introduced a variant of edge domination in chapter 2,  
edge covering in chapter 4 and edge independence in chapter 5 from several viewpoints  
including vertex removal, edge removal and edge addition to the graph.

Hypergraphs are natural extensions of graphs in which edges may consist of more  
than 2 vertices. The hypergraph is an order pair $G = (V, E)$. The nonempty set $V$  
contains the elements $\{v_1, v_2, \ldots , v_m\}$ and $E = \{C_1, C_2, \ldots , C_m\}$ is a family of nonempty  
subsets of $V$ such that $\bigcup_{i=1}^{m} C_i = V$ [4]. The elements of $V$ and $E$ are called the vertices  
and the edges of the hypergraph $G$ respectively.

If the hypergraph $G$ is clear from the context, we simply write $V = V(G)$ and $E  
= E(G)$. We shall use the notations $n = |V|$ and $m = |E|$ to denote the order and  
size of the hypergraph $G$, respectively. If an edge $E_1 \in E$ then $|E_1|$ is the size of an edge $E_1$. An  
isolated edge of the hypergraph $G$ is an edge that does not intersect any other edge of $G$.

We make the following conventions about the hypergraph considered throughout  
the work unless otherwise stated.

(1) If $x$ and $y$ are distinct vertices then there is atmost one edge which contains $x$ and $y$.  
(2) Any two distinct edges of the hypergraph will intersects in atmost one vertex.  
(3) If $e$ and $f$ are edges having atleast two vertices then $e \not\subseteq f$ and $f \not\subseteq e$.

Let $v$ be a vertex of the hypergraph $G$ such that $\{v\}$ is not an edge of the hyper-
graph $G$ then the sub hypergraph $G - v$ of $G$ is a hypergraph whose vertex set is $V -  
\{v\}$ and the edge set is equal to $\{ e' \mid e' \text{ is nonempty and } e' = e - \{v\}, e \in E(G)\}$ [4]. A partial sub hypergraph $G - v$ of $G$ is a hypergraph with vertex set $V - \{v\}$ and the  
edge set equal to $\{e \in E(H) \mid v \notin e \}$.  

Two vertices $x$ and $y$ of the hypergraph $G$ are adjacent if there is an edge $E_1$ of $G$  
such that $\{x, y\} \subseteq E_1$.

A set of vertices $S \subseteq V(G)$ of the hypergraph $G$ is a dominating set of $G$ if for  
each vertex $v \in V(G) - S$, there is a vertex $u \in S$ such that the vertices $u$ and $v$ are  
adjacent in $G$ [2]. A set of vertices $S \subseteq V(G)$ is an H-dominating set of the hypergraph  
$G$ if for each vertex $v \in V(G) - S$, there is an edge $F$ containing $v$ such that $F - \{v\}$  
is a nonempty subset of $S$ [14].
If $G = (V(G), E(G))$ is a graph without isolated vertices then the dual hypergraph of the graph $G$ (denoted by $G^*$) is a hypergraph with vertex set $V(G^*) = E(G)$ and edge set $E(G^*) = \{ \overline{v} \mid v \in V(G) \}$, where $\overline{v} = \{ e \in E(G) \mid v \text{ is an end vertex of } e \}$ [4].

We have considered the hypergraph in chapter 2 with the concept edge domination in hypergraph. A new concept called an edge H-domination in graph is defined in chapter 3. An algorithm to find an edge H-dominating set of the graph is written. It is also proved that the edge H-dominating set obtained by an algorithm is minimal. The change in edge H-domination number of the graph is observed under the operations like vertex removal and edge removal in this chapter.

A subset $T$ of vertices in a hypergraph $G$ is a transversal (also called vertex cover) in $G$ if $T$ has a nonempty intersection with every edge of $G$ [4]. The transversal number of $G$ is the minimum size of a transversal in $G$. The total transversal of the hypergraph $G$ is a transversal $T$ with the property that every vertex in $T$ has at least one neighbor in $T$ [12].

We have defined the total transversal in graph $G$ [23]. The total transversal of the graph is a vertex cover without isolated vertices. The minimum cardinality of a total transversal of the graph $G$ is known as the total transversal number (denoted by $\alpha_{tt}(G)$) of the graph $G$. The bounds on the edge domination number of the graph are obtained in terms of total transversal number in chapter 2 and these bounds are improved with the help of edge cover of the graph in chapter 4.

**Definition 1.1.** A set $S \subseteq V$ is said to be an extended strong vertex cover of the hypergraph $G$ if the following two conditions hold.

1. If $\{ v \}$ is an edge then $v \in S$.
2. If $u$ and $v$ are adjacent vertices of $G$ then $u \in S$ or $v \in S$.

**Proposition 1.2.** Every extended strong vertex cover of the hypergraph is a vertex cover.

**Proof.** Let $S$ be an extended strong vertex cover and $e$ be any edge of the hypergraph $G$. If $e = \{ v \}$ then $v \in S$. Therefore $e \subseteq S$ and thus $e \cap S \neq \emptyset$. If $e$ contains more than one vertices. Let $u, v \in e$ with $e \cap S = \emptyset$. Thus $u$ and $v$ are adjacent vertices but $u \notin S$ and $v \notin S$. Which is a contradiction. Therefore $e \cap S \neq \emptyset$. Thus $S$ is a vertex cover. □
A new concept called a strong edge cover of the graph is defined in chapter 4. The relations between edge cover and strong edge cover also between edge H-domination and strong edge cover are discussed in this chapter.

**Definition 1.3.** [4] Let $G = (V, E)$ be a hypergraph. A set $S \subseteq V$ is said to be a **stable set** if $S$ does not contain any edge $F$ such that $|F| > 1$. The **stability number** (denoted by $\alpha(G)$) of the hypergraph $G$ is defined as the maximum cardinality of a stable set in $G$.

**Definition 1.4.** [4] Let $G = (V, E)$ be a hypergraph. A set $S \subseteq V$ is said to be an **independent set** if no two vertices of $S$ are adjacent. The maximum cardinality of an independent set in a hypergraph $G$ is called the **independence number** (denoted by $\beta(G)$) of the hypergraph $G$.

An independent set of the hypergraph is also known as "strongly stable set". It is obvious that the set of all maximal stable sets in a hypergraph $G$ contains the set of all maximal independent sets in $G$. it follows that, $\alpha(G) \geq \beta(G)$, for any hypergraph $G$.

The edge independent set is considered in chapter 5. The change in edge independence number is observed under the operation edge removal from the graph. A new concept called an edge stability in graph is defined in chapter 6. A relation between edge H-domination and edge stability is discussed. The change in the edge stability number of the graph is observed under edge removal operation.

**Definition 1.5.** A set $S \subseteq V$ is said to be an **extended independent set** of the hypergraph $G$ if the following two conditions hold.

1. If $\{v\}$ is an edge then $v \notin S$.
2. If $u$ and $v$ are distinct vertices of $S$ then $u$ and $v$ are non-adjacent.

**Proposition 1.6.** Every extended independent set of the hypergraph is a stable set.

**Proof.** Let $S$ be an extended independent set of the hypergraph $G$ and $S$ contains an edge $e$. Suppose that $e = \{v\}$ then $v \in S$, which contradicts the definition of extended independent set. Suppose that $e$ contains more than one vertices. Let $u, v \in e$ then assume that $u, v \in S$ and $u$ and $v$ are adjacent vertices, which again contradicts the definition of extended independent set. Thus $S$ must be a stable set. \[\square\]
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A new concept called a strong edge independence in graph is defined in chapter 5. The relation between strong edge cover and strong edge independence of the graph is discussed in this chapter. The change in the strong edge independence number is observed under the operation edge removal from the graph.

The concluding remarks of the work are written in chapter 7. In this chapter, we have given some possible new directions through which this research can be extended further.