CHAPTER[1] 1: EXPONENTIAL MATRICES AND THEIR PROPERTIES

1.1- Introduction

The exponential Matrices have been widely applied in the field of science and especially in matrix analysis so, the exponential matrix has been given a special concern as it is a very significant particular of matrix functions. This chapter deals with the definition of the matrix exponential and its general properties. The main focus of this chapter is on the matrix exponential. The theory of the matrix exponential was subsequently developed by many mathematicians over the last 100 years. Today, the matrices exponential are broadly used in science and engineering. There is growing interest due to the brief way they allow solutions to be expressed and to the recent advances in numerical algorithms for computing them [16,20]. Generally, there are interesting areas in linear algebra and the matrix exponential has been one of these interesting areas.

1.2- Definition of Exp (A)

The exponential function of matrices is a very important a particular matrix of functions of matrices so, the exponential of A can be defined in several (different) ways. In mathematics, the exponential matrix is a function on square matrices analogous to the ordinary

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exponential function [1,16,20]. Let $A \in M^{n\times n}$ the exponential of A, denoted by e^A or $\exp(A)$, is the $n\times n$ matrix given by the power series

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!} = I + A + \frac{A^2}{2!} + ... + \frac{A^{n-1}}{(n-1)!} + ... \tag{1.1}$$

Where $A^0 = I$.

The above series (1.1) always converges absolutely for all A, so the exponential of A is well-defined. Note that this is the generalization of the Taylor series expansion of the standard exponential function. To prove the Convergence of the series, we have the following theorem.

Theorem (1.1) [16,20]

The series (1.1) converges absolutely for all $A \in M^{n\times n}$. Moreover, let $\|\cdot\|$ be a normalized sub multiplicative norm on $M^{n\times n}$. Then

$$\left\|e^A\right\| \le e^{\|A\|} \tag{1.2}$$

Proof:

The n^{th} partial sum is $S_n = \sum_{k=0}^{\infty} \frac{A^k}{k!}$

So, $\left\|e^A - S_n\right\| \equiv \left\|\sum_{k=0}^{\infty} \frac{A^k}{k!} - \sum_{k=0}^{m} \frac{A^k}{k!}\right\| = \left\|\sum_{k=m+1}^{\infty} \frac{A^k}{k!}\right\| \le \sum_{k=m+1}^{\infty} \frac{\left\|A^k\right\|}{k!} \le \sum_{k=m+1}^{\infty} \frac{\|A\|^k}{k!}$

Since $\|A\|$ is a real number and the right-hand side is a part of the convergent series of real numbers

$$e^{\|A\|} = \sum_{k=0}^{\infty} \frac{\|A\|^k}{k!}$$

Then this equation is convergent, if $\varepsilon > 0$ there is an N such that for $m > n$,

$$e^{\|A\|} = \sum_{k=m+1}^{\infty} \frac{\|A\|^k}{k!} < \varepsilon$$

This is sufficient to prove that S_n is convergent. Furthermore, note that

$$\|e^A\| = \left\|\sum_{k=0}^{\infty} \frac{A^k}{k!}\right\| \leq \sum_{k=0}^{\infty} \frac{\|A^k\|}{k!} \leq \sum_{k=0}^{\infty} \frac{\|A\|^k}{k!} = e^{\|A\|}$$

In some cases, it is easy to illustrate the matrix exponential of $n \times n$ complex matrix A. It will be denoted by e^A and can be defined in a number of equivalent ways:

$$e^{At} = \frac{1}{2\pi i} \oint_{\Gamma} e^{zt} (zI - A)^{-1} dz \tag{1.3}$$

Or

$$e^{At} = lim_{k \to \infty} (1 + \frac{At}{k})^k \tag{1.4}$$

Or

$$e^{At} \Leftrightarrow \frac{dx}{dt} = AX(t) \ , \quad X(0) = 1 \tag{1.5}$$

For more details see [17,20,33].

1.3- Properties

In this section of this chapter, additional significant properties of the exponential matrix which do not require further development are collected [10,18,19,33]. Let $A, B \in M^{n\times n}$ and let t and s be arbitrary complex numbers. The $n\times n$ Zero matrix is denoted by 0. The exponential matrix satisfies the following properties.

- Property (1) If 0 denotes the zero matrix, then $e^{0} = I$ the identity matrix.
- Property (2) *If* A *is invertible, then* $e^{ABA^{-1}} = Ae^{B}A^{-1}$.
- Property (3)
  if $A = diag(A_1, A_2, ..., A_k)$, *then* $e^{A} = (e^{A_1}, ..., e^{A_k})$.
- Property (4) $det(e^{A}) = e^{trac(A)}$ when A is complex square matrix and trace(A)=0 then $det(e^{A}) = 1$.
- Property (5) $e^{(A^{T})} = (e^{A})^{T}$ follows that if A is symmetric, then e^{A} is also symmetric, and if A is skew symmetric then e^{A} is orthogonal.
- Property (6) if $AB = BA$ *then* $Ae^{B} = e^{B}A$ *and* $e^{A}e^{B} = e^{B}e^{A}$ unfortunately, not all familiar properties of the scalar exponential function $y = e^{t}$ carries over to the exponential matrix. For example, we know from calculus $e^{s+t} = e^{s}e^{t}$ when s and t are numbers. However, this is often not true for exponentials of matrices. In other words, it is possible to have $n\times n$ matrices A and B such that $e^{A+B} \neq e^{A}e^{B}$. Exactly

when we have equality $e^{A+B}=e^{A}e^{B}$ depends on specific properties of the matrix A and B that discussed in this section.

- Property (7) let A be a complex square $n\times n$ Matrix, then $A^{m}e^{A}=e^{A}A^{m}$ *for integer m.*
- Property (8) Let A be a complex square matrix, and let $s,t\in C,$ Then $e^{A(s+t)}=e^{As}e^{At}$.

Setting $s=1$ and $t=1$ in property (1) we get $e^{0}=I$. In other words, regardless of the matrix A, the matrix exponential e^{A} is always invertible, and has inverse e^{-A}.

- Property (9) $e^{(A^{*})}=(e^{A})^{*}$. It follows that if A is Hermitian matrix, then e^{A} is also Hermitian, and if A is skew-hermitian, then e^{A} is.
- Property (10) $(e^{At})'=Ae^{tA}$.
- Property (11) let $A,B\in M^{n}$ then $AB=BA$ if and only if for all t such that $e^{(A+B)t}=e^{At}e^{Bt}$.
- Property (12) let $A,B\in M^{n}$ be given. If $AB=BA$, then $e^{(A+B)}=e^{A}e^{B}=e^{B}e^{A}$.

1.4 - Proofs and Remarks

In this section, we discuss some proofs and some remarks on the previous section, and we will give some examples related to them. For property (1) it is easy to prove it. For property (2), we prove it. Recall that, for all integers $s\geq 0$, we have

$$(ABA^{-1})^m = AB^m A^{-1}$$

by using the definition (1) to get

$$\begin{aligned} e^{ABA^{-1}} &= I + ABA^{-1} + \frac{(ABA^{-1})^2}{2!} + \ldots\ldots\ldots \\ &= I + ABA^{-1} + A\frac{B^2}{2!}A^{-1} + \ldots\ldots\ldots \\ &= A(I + B + \frac{B^2}{2!} + \ldots\ldots\ldots)A^{-1} = Ae^B A^{-1} \end{aligned}$$

But If a matrix B is a diagonalizable matrix, then there will exist an invertible A so that $B = ADA^{-1}$, where D is a diagonal matrix of eigenvalues of B, and A is a matrix having eigenvectors of B as its columns. In this case $e^B = Pe^D P^{-1}$. For property (3) it is easy to prove it. For property (4), If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $A \in n \times n$, then $(e^{\lambda_1}, e^{\lambda_2}, \ldots\ldots, e^{\lambda_k})$ are the eigenvalues of e^A by the spectral mapping property for diagonalizable matrices such that $f(A) = Pf(A)P^{-1}$ where P is an invertible matrix.

Then the trace is the sum of the eigenvalues, and the determinant is the product of the eigenvalues of A, so

$$det(e^A) = (e^{\lambda_1} e^{\lambda_2} \ldots\ldots e^{\lambda_n}) = e^{\lambda_1} + e^{\lambda_2} + \ldots.. + e^{\lambda_n} = e^{trac(A)}.$$

For Property (5), it is easy if A is symmetric such that $A = A^T$ then by using the definition of e^A that

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}$$

Then,

$$e^{A^T} = \sum_{k=0}^{\infty}\frac{(A^T)^k}{k!} = \sum_{k=0}^{\infty}\frac{(A^k)^T}{k!} = (\sum_{k=0}^{\infty}\frac{A^k}{k!})^T = (e^A)^T \quad .$$

For property (6), we use the definition of e^A then

$$Ae^B = A\sum_{k=0}^{\infty}\frac{B^k}{k!} = A(lim_{n\to\infty}\sum_{k=0}^{n}\frac{B^k}{k!}) = lim_{n\to\infty}\sum_{k=0}^{n}\frac{AB^k}{k!}$$

$$= lim_{n\to\infty}\sum_{k=0}^{n}\frac{B^kA}{k!} = (\sum_{k=0}^{\infty}\frac{B^k}{k!})A \quad by\,(AB = BA)$$

Then we can use induction to prove property (7), from property (6) to get

$$A^m e^A = e^A A^m .$$

For property (8), we use the definition of e^A then we have

$$\begin{aligned} e^{As}e^{At} &= (I + As + \frac{A^2s^2}{2!} + \frac{A^3s^3}{3!} + ...)(I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + ...) \\ &= (\sum_{j=0}^{\infty}\frac{A^js^j}{j!})\ (\sum_{k=0}^{\infty}\frac{A^kt^k}{k!}) \\ &= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{A^{j+k}s^jt^k}{j!k!} \end{aligned}$$

Put $m = j + k$, then $j = m - k$ then from the binomial theorem that

$$e^{As}e^{At} = \sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\frac{A^ms^{m-k}t^k}{(m-k)k!} = \sum_{m=0}^{\infty}\frac{A^m}{m!}\sum_{k=0}^{\infty}\frac{m!}{(m-k)!k!}s^{m-k}t^k =$$

$$= \sum_{m=0}^{\infty}\frac{A^m(s+t)^m}{m!} = e^{A(s+t)}$$

Property (9) is as similar as property (5) and property (10). It is easy to be proved. Property (11), if $AB = BA$ and by using the power series expansion of $e^{At}e^{At}$ and $e^{(A+B)t}$,it will be commuted and identical. On the other hand, if $e^{(A+B)t} = e^{At}e^{Bt}$ for all t and by differentiating it twice with respect to t and put $t = 0$, the result will be $AB = BA$. The property (12) is the most important in matrix exponential. So, it can be proved as following:

The power series for e^{A} is used to prove this property as follows

$$e^{A}e^{B} = (I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + ...)(I + B + \frac{B^2}{2!} + \frac{B^3}{3!} + ...)$$

$$= (\sum_{j=0}^{\infty}\frac{A^j}{j!})\ (\sum_{k=0}^{\infty}\frac{B^k}{k!})$$

$$= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(A+B)^{j+k}}{j!\,k!}$$

Put $m = j + k$, then $j = m - k$ then from the binomial theorem that

$$e^{A}e^{B} = \sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\frac{A^m B^{m-k} t}{(m-k)k!}$$

$$= \sum_{m=0}^{\infty}\frac{A^m}{m!}\sum_{k=0}^{\infty}\frac{m!}{(m-k)!\,k!}B^{m-k}$$

$$= \sum_{m=0}^{\infty}\frac{(A+B)^m}{m!} = e^{(A+B)}$$

It then follows that $e^{A}e^{B} = e^{(A+B)} = e^{B+A} = e^{B}e^{A}$.

To conclude, although commutativity is a sufficient condition for the identities

$$e^{A}e^{B}=e^{(A+B)}=e^{B+A}=e^{B}e^{A}$$

to hold, but it is not necessary to hold as we see to the example

$$A=\begin{pmatrix}0 & 0\\ 0 & 2\pi i\end{pmatrix}, B=\begin{pmatrix}0 & 1\\ 0 & 2\pi i\end{pmatrix}$$

Shows

$$(e^{(A+B)}=e^{A}=e^{B}=I).$$

If A and B have algebraic entries, then their commutativity is necessary for

$$e^{A}e^{B}=e^{(A+B)}=e^{B+A}=e^{B}e^{A}$$

to hold [16,17,20,33].

1.5-Usefulness

The exponential of a matrix is a very important function can be used in various fields. One of the reasons for the importance of the matrix exponential is that it can be used to solve systems of linear ordinary differential equations. Another use of the matrix exponential is that mathematical models of many physical, biological, and economic processes involve systems of linear ordinary differential equations with constant coefficient,

$$x'=Ax(t)$$

Here $A\in M^{n\times n}$ is a given fixed matrix. A solution vector $x(t)$ is sought, which satisfies an initial condition

$$x(0)=X_{0}.$$

In control theory, A is known as the state companion matrix and $x(t)$ is the system response. In principal, the solution is given by

$$x(t) = e^{tA} X_0$$

Where e^{tA} can be formally defined by the convergent power series

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!} = I + A + \frac{A^2}{2!} + ... + \frac{A^{n-1}}{(n-1)!} + ...$$

The matrix exponential can also be used to solve the inhomogeneous equation

$$x'(t) = Ax(t) + z(t), x(0) = X_0 .$$

1.6-Literature Review

The exponential matrix has a major role in scientific fields particularly in matrix analysis, linear systems and control theory. A particular attention is paid to previous work related to the present research. In this section, explicit formulas for computing the exponential of some special matrices are given. These formulas are derived for the exponential of 2×2 complex matrix A.

Formulae are given in terms of either the eigenvalues of A or the entries of A. The results are specialized to the case in which A is a real matrix. Let R and C denote the real and complex numbers respectively. The objective in this section is to give explicit formulas to simplify its exposition and usage. This section will collect together as many such formulae as possible in one place. In addition to their usefulness in linear system theory, these formulae should be helpful in future research concerning the matrix exponential. In this section, we begin with deriving formulas for the exponential of an arbitrary 2×2 matrix in terms of either its eigenvalues or entries. These results are then used in the second-order mechanical vibration equation with weak or strong damping. Here, also, we derive some formulas for the exponential of $n\times n$ matrices which are given for matrices that satisfy an arbitrary quadratic polynomial. Beside the 2×2 matrices, we derive nilpotent and idempotent matrices as well [22, 33].

1.5.1-THE EXPONENTIAL OF A GENERAL 2×2 MATRIX

Lemma (1.5.1.1):

Let $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in C^{2\times 2}$ then,

1-if $a=d$ then $e^A=e^a\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$,

2-if $a\neq d$ then $e^A=\begin{pmatrix} e^a & b\dfrac{(e^a-e^d)}{a-d} \\ 0 & e^d \end{pmatrix}$,

The following theorem explains the general case of 2×2 by using the eigenvalues of A.

Theorem (1.5.1.2):

Let λ_1 and λ_2 denote the eigenvalues of $A\in C^{2\times 2}$ then

1-if $\lambda_1=\lambda_2$ then $e^A=e^{\lambda}[(1-\lambda)I+A]$,

2-if $\lambda_1\neq\lambda_2$, then $e^A=\dfrac{\lambda_1 e^{\lambda_2}-\lambda_2 e^{\lambda_1}}{\lambda_1-\lambda_2}I+\dfrac{e^{\lambda_1}-e^{\lambda_2}}{\lambda_1-\lambda_2}A$.

Lemma (1.5.1.3):

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in C^{2\times 2}$ then,

1-if

$$(a-d)^2 + 4bc = 0, then$$
$$e^A = e^{\frac{(a-d)}{2}} \begin{pmatrix} 1 + \frac{a-d}{2} & b \\ c & 1 - \frac{a-d}{2} \end{pmatrix}$$

2-if

$$(a-d)^2 + 4bc \neq 0, then$$
$$e^A = e^{\frac{(a-d)}{2}} \begin{pmatrix} \cosh\Delta + \frac{a-d}{2}\frac{\sinh\Delta}{2} & b\frac{\sinh\Delta}{\Delta} \\ c\frac{\sinh\Delta}{\Delta} & \cosh\Delta - \frac{a-d}{2}\frac{\sinh\Delta}{\Delta} \end{pmatrix}$$

Where, $$\Delta = \frac{1}{2}\sqrt{(a-d)^2 + 4bc.}$$

Lemma (1.5.1.4):

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in R^{2\times 2}$ then

1-if

$$(a-d)^2+4bc=0, then$$

$$e^A=e^{\frac{(a+d)}{2}}\begin{pmatrix} 1+\frac{a-d}{2} & b \\ c & 1-\frac{a-d}{2} \end{pmatrix}$$

2-if

$$(a-d)^2+4bc>0, then$$

$$e^A=e^{\frac{(a+d)}{2}}\begin{pmatrix} cosh\delta+\frac{a-d}{2}\frac{sinh\delta}{2} & b\frac{sinh\delta}{\delta} \\ c\frac{sinh\delta}{\delta} & cosh\delta-\frac{a-d}{2}\frac{sinh\delta}{\delta} \end{pmatrix}$$

Where $\delta=\frac{1}{2}\sqrt{(a-d)^2+4bc.}$

3-if

$$(a-d)^2+4bc<0, then$$

$$e^A=e^{\frac{(a+d)}{2}}\begin{pmatrix} cos\delta+\frac{a-d}{2}\frac{sin\delta}{2} & b\frac{sin\delta}{\delta} \\ c\frac{sin\delta}{\delta} & cos\delta-\frac{a-d}{2}\frac{sin\delta}{\delta} \end{pmatrix}$$

Where $\delta=\frac{1}{2}\sqrt{\left|(a-d)^2+4bc\right|}.$

1.5.2 -THE EXPONENTIAL OF $N \times N$ MATRICES SATISFYING A QUADRATIC POLYNOMIAL

In this section, we derive formulas for $n \times n$ matrices that satisfy a quadratic polynomial. Since, by the Cayley-Hamilton Theorem, this section contains all 2×2 matrices, the results of the previous section are recovered as a special case. In addition, these results are applied to certain $n \times n$ matrices such as involutory and idempotent matrices [22,33].

Lemma 1.5.2.1:

Let $A \in C^{n \times n}$ and suppose that $A^2 = \rho I$, where $\rho \in C$, then

1-if $\rho = 0 \Rightarrow e^A = I + A$.

2-if $\rho \neq 0 \Rightarrow e^A = cosh(\sqrt{\rho})I + (\frac{sinh\sqrt{\rho}}{\sqrt{\rho}})A.$

Lemma (**1.5.2.1**) applies to nilpotent and involutory matrices.

Corollary (1.5.2.2):

Let $A \in C^{n \times n}$ thenو

1-if $A^2 = 0 \Rightarrow e^A = I + A$

2-if $A^2 = I \Rightarrow e^A = cosh(1)I + sinh(1)A$.

Theorem (1.5.2.3):

Let $A \in C^{n \times n}$ and suppose that $A^2 + 2\lambda_1 A + \lambda_2 I = 0,\ where\ \lambda_1, \lambda_2 \in C$, then

1-if $\lambda_1^2 = \lambda_2$, then $e^A = e^{-\lambda_1}[(1 + \lambda_1)I + A]$

2-if $\lambda_1^2 > \lambda_2, then$

$$e^A = e^{-\lambda_1}\Big\{[cosh(\sqrt{\lambda_1^2 - \lambda_2}) + \frac{\lambda_1 \, sinh(\sqrt{\lambda_1^2 - \lambda_2})}{\sqrt{\lambda_1^2 - \lambda_2}}]I + \frac{1}{\sqrt{\lambda_1^2 - \lambda_2}} sinh(\sqrt{\lambda_1^2 - \lambda_2})A]\Big\}$$

3-if $\lambda_1^2 < \lambda_2, then$

$$e^A = e^{-\lambda_1}\Big\{[cos(\sqrt{\lambda_2 - \lambda_1^2}) + \frac{\lambda_1 \, sin(\sqrt{\lambda_2 - \lambda_1^2})}{\sqrt{\lambda_2 - \lambda_1^2}}]I + \frac{1}{\sqrt{\lambda_2 - \lambda_1^2}} sin(\sqrt{\lambda_2 - \lambda_1^2})A]\Big\} .$$

and every chapter has literature review for all previous studies which is related to work as we see in the next chapters.