# CHAPTER 7: CONCLUSION AND FUTURE WORK

## 7.1- CONCLUSION

This chapter is the conclusion of the thesis. It summarizes the main ideas discussed in every chapter. There are many ways to define $e^A$, leading to many algorithms to compute $e^A$. The first part of the conclusion deals with the question: "which method is the best?" To answer such a question is very risky, because we have no enough knowledge about the sensitivity of the original problem, or about the detailed performance of the careful implementations of various methods to make any firm conclusions. In short, we have to look at the points of strength and weakness of each method.

Using the polynomial methods does not mean they are really in the competition for "best". Some of them require the characteristic polynomial and so they are appropriate only for certain special problems. While, others have the same stability difficulties as matrix decomposition methods, but are much less efficient. We have seen that when we calculated the matrix exponential by the Jordan form. This way is very boring for a big matrix in size, especially when the matrix is defective. It is also difficult to be determined numerically because any small changes in a defective matrix will completely change the Jordan form. However, when we use Taylor's Series numerically, which is always theoretical converge; we will have large cancelation errors due to truncated Taylor's series. Therefore, the convergence can be slow, if $\|A\|$ is large. We can avoid such a problem by careful

Scaling and Squaring method. Eigenvalues- Eigenvectors method does not work when $A$ is not diagonalizable. Also, we have a problem related to$A$ anon-diagonalizable matrix. Another problem would be attributed to Pade approximation. This method does not always converge. However, with suitable Scaling and squaring it always converges. Finally, we have seen that there is no uniformly best method for the computation matrix exponential. The choice of method depends on the details of implementation and the specific problems being solved.

The second part of conclusion presents and introduces a new method to compute the exponential matrix where this matrix in $M(2,R)$. Some lemmas and theorems that explain the procedure of this method have been presented in this part. The researcher shall also determine the necessary and sufficient conditions for the two matrices in $M(2,R)$ to satisfy the equation $e^{A+B} = e^{A}e^{B} = e^{B}e^{A}$, depending on the new method. The result of Theorem (3) can also be expressed in topological terms, noting that the center $C(2,R)$, which is a lie group is not connected, obviously isomorphic to the multiplicative group $\{R^{+}, X\}$. Its two components are connected sets of matrices scalar positive and negative. Theorem (3) can then be reformulated by saying that $e^{A+B} = e^{A}e^{B}$ if they belong to the same connected component of $C(2,R)$.It is not difficult to extend these result so the case of the two matrices in $M(2,R)$, just note that its center $C(2,R)$ is connected. Then, in Theorem (3) just note that $e^{A+B}$ , $e^{A}e^{B}$ , $e^{B}e^{A}$ are in the center. On the other hand, Theorem (4) is no longer valid, because the function $w=(e^z-1)/z$ periodic in C and therefore $M(2,C)$ are infinite matrices such as $e^{A+B}=e^Ae^B\neq e^Be^A$. From the first search in literature, it does not seem to be a solution in the general case of matrices $M(n,K)$ with $K=R\ or\ C$, even in the case n = 3. Also, we construct a new method to compute $e^A$ without using eigenvalues, so this method is different. This is because the previous methods need to calculate the eigenvalues of matrix $A$ to compute $e^A$, as the method of canonical Jordan form. It is more complicated, when we use a bigger matrix than in this method that calculating the coefficients of series $g_k$.we also need to do multiplication and division of numbers that can easily be obtained to calculate the coefficients. In contrast, the method of canonical Jordan form must calculate the eigenvalues, eigenvectors, the kernels of the matrices $(A-\lambda I)^k$ and the inverse matrix $P$. However, if the matrix is large, say, size is 20, and then the calculations to canonical Jordan form are more complicated than this method. Although we have applied the formula (16) for the exponential function, it is general and it can calculate matrices and other functions as in $ln(A)$. In order to increase the accuracy, we can cut the series $g_k$ after some terms. Then, we can ask whether there will be any way to write the formula (16) such as taking a few terms in $g_k$ good accuracy.

Also, there are many ways to compute the exponential matrix. The current study provides a new method to compute the exponential matrix in which this matrix is located in $M(2,R)$. The study also presents some lemmas and theorems that explain the procedure of this method. Based on the previous part, the problem of determining the necessary and sufficient conditions for two matrices $M(2,R)$by verifying the equation $e^{A+B} = e^A e^B = e^B e^A$ is thus entirely solved. The result of Theorem (5.3) can also be expressed in topological terms, noting that the center $C(2,R)$ which is a Lie group is not connected, obviously isomorphic to the multiplicative group $\{R^+, X\}$ and its two components are connected sets of scalar positive and negative matrices. Theorem (5.3) can then be reformulated by saying that $e^{A+B} = e^A e^B$ if they belong to the same connected component of$C(2,R)$. It is not difficult to extend these results to the case of two the matrices in $M(2,R)$, just note that the its center $C(2,R)$ is connected, then in Theorem (5.3) just let that $e^{A+B}$ , $e^A e^B$ , $e^B e^A$ by are in the center, while Theorem (5.4) is no longer valid because the function $w = (e^z - 1)/z$ periodic in C and therefore $M(2,C)$there are infinite matrices such as $e^{A+B} = e^A e^B \neq e^B e^A$.From a first search in the literature it does not seem to be a solution in the general case of matrices $M(n,K)$ with$K = R \; or \; C$, even in the case n = 3.

Finally, in this chapter we presented new method to compute the exponential matrix $e^{At}$ as accurate solution. The basic idea of this method is based on the matrix theory; the matrices satisfy the special case

$$A^n = \omega_1 A^{n-1} + \omega_2 A^{n-2} + \omega_3 A^{n-3} + ..... + \omega_k A^{n-k}, k < n.$$

Furthermore, this method can be extended to the more general case

$$A^n = \omega_1 A^{k-1} + \omega_2 A^{k-2} + \omega_3 A^{k-3} + ..... + \omega_k A^{k-m}, k < n, m < k.$$