CHAPTER[6] 6: NEW APPROACH TO CALCULATE THE EXPONENTIAL MATRIX FOR SOME SPECIAL MATRICES

6.1-Introduction

The exponential matrix e^{tA} is a very important a particular matrix functions. It is a very useful tool for solving linear systems, which help us in the analysis of controllability and observability of a linear system and control theory.

Basically, matrix functions are widely used in science area especially in matrix analysis. It provides a formula for closed solutions, which help us in the analysis of controllability and observability of a linear system [23,50]. There are several methods for calculating the exponential matrix, some of these methods are effective and others not effective. The differential equation

$$x'(t) = Ax(t)$$

has the solution

$$e^{tA}x(0),$$

which plays an important role in linear system and control theory. It is well known that the exponential matrix e^{At} can be defined by a convergent power series: $e^{tA} = \sum_{n=0}^{\infty} \frac{(tA)^n}{n!}$

However, calculating the powers of matrix A which is an infinite sum ,

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makes the researchers design methods to get the accurate solution of e^{tA}. In recent years, there exist many methods for computing e^{tA} for more details [10,39,48,49,51,52]. Among these methods, for example, the explicit formulae can overcome the truncation errors which are widely used in these papers [39, 48, 49, 51, 52], which are based on this paper [39].

In this chapter, we introduce new method to compute the matrix exponential e^{tA} where the matrix $A \in C^{n\times n}$ has an eigenvalues $\lambda_1 = 0$. This method comput**e** the accurate solution of e^{tA} where the matrix $A \in C^{n\times n}$ satisfy this condition

$$A^n = \omega_1 A^{n-1} + \omega_2 A^{n-2},$$

and ω_1 *and* ω_2 are parameters. When

$$\omega_1 = 0 \text{ or } \omega_2 = 0,$$

then the matrix A is the same as the matrix in [48]. In the last part of this chapter, we provide examples to show the effectiveness of this method.

6.2- The Main Results

It is well known that the exponential matrix e^{At} can be defined by a convergent power series

$$e^{tA} = \sum_{n=0}^{\infty} \frac{(tA)^n}{n!} = I + tA + \frac{(tA)^2}{2!} + \ldots\ldots + \frac{(tA)^{n-1}}{(n-1)!} + \ldots\ldots\ldots\ldots \qquad (6.1)$$

Some authors and researchers gave explicit formulae to compute the exponential matrix, for example, Bernstein and So in his papers

[39,51] found a new method to compute the exponential matrix e^{tA} when

$$A^2 = A\ , A^2 = \omega\, I_n \ and \ \ A^3 = \omega A$$

and Wu, B. B. [48] found a new method to compute the exponential matrix e^{tA} when:

$$A^{n+1} = \omega A^n\ , A^{n+2} = \omega^2 A^n \ and \ \ A^{n+3} = \omega^3 A^n\ .$$

In this chapter, we will generalize a method which presented in Wu, B. B. [48] to compute the exponential matrix e^{At} when $A \in C^{n\times n}$ has this form

$$A^n = \omega_1 A^{n-1} + \omega_2 A^{n-2} + \omega_3 A^{n-3} + ... + \omega_k A^{n-k}\ , k < n\ .$$

Suppose

$$A^n = A^{n-1} + A^{n-2}\,,$$

then

$$A^{n+1} = 2A^{n-1} + A^{n-2}$$

Continue this process until we get

$$A^{n+k} = \alpha_1 A^{n-1} + \alpha_2 A^{n-2}\,,$$

by using this formulae, we rewrite equation (6.1) as follow

$$e^{At} = I + tA + \frac{t^2A^2}{2!} + ... + \alpha_1(t)A^{n-2} + \alpha_2(t)A^{n-1}\ . \tag{6.2}$$

To compute the parameters $\alpha_1(t),\alpha_2(t)$ as the following process:

From equations (6.1,6.2) and putting

$$A^n = A^{n-1} + A^{n-2},\ \ A^{n+1} = 2A^{n-1} + A^{n-2}$$

and so on in all terms and collect them in the same kind we get

$$I + \frac{tA}{1!} + ... + \frac{t^{n-2}A^{n-2}}{(n-2)!} + \frac{t^{n-1}A^{n-1}}{(n-1)!} + ... = I + \frac{tA}{1!} + ... + \alpha_1(t)A^{n-2} + \alpha_2(t)A^{n-1}$$

By comparing all factors we get

$$\alpha_1(t) = \frac{t^{n-2}}{(n-2)!} + \frac{t^n}{(n)!} + \frac{t^{n+1}}{(n+1)!} + \frac{2t^{n+2}}{(n+2)!} + \ldots.. \tag{6.3}$$

$$\alpha_2(t) = \frac{t^{n-1}}{(n-1)!} + \frac{t^n}{(n)!} + \frac{2t^{n+1}}{(n+1)!} + \frac{3t^{n+2}}{(n+2)!} + \ldots.. \tag{6.4}$$

By looking to equations (6.3) and (6.4) we see that infinite series, so it is difficult to find the exact form for $\alpha_1(t), \alpha_2(t)$, which we will discuss them in next section.

Definition (1):

A matrix

$$\begin{bmatrix} 1 & s_1 & s_1^2 & \cdots & s_1^{n-1} \\ 1 & s_2 & s_2^2 & \cdots & s_2^{n-1} \\ 1 & s_3 & s_3^2 & \cdots & s_3^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & s_n & s_n^2 & \cdots & s_n^{n-1} \end{bmatrix}$$

is called Vandermonde matrix, and

$$\begin{vmatrix} 1 & s_1 & s_1^2 & \cdots & s_1^{n-1} \\ 1 & s_2 & s_2^2 & \cdots & s_2^{n-1} \\ 1 & s_3 & s_3^2 & \cdots & s_3^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & s_n & s_n^2 & \cdots & s_n^{n-1} \end{vmatrix} = \prod_{1 \le j < i \le n} (s_i - s_j)$$

is the determinant of the Vandermonde matrix .

Definition (2):

The characteristic polynomial of the matrix $A \in C^{n\times n}$ is $f(\lambda)$, then

$$f(A) = A^n - c_{n-1}A^{n-1} - c_{n-2}A^{n-2} - .... - c_1 I = 0$$

By applying definition (2) to equation (6.1) we have the following equation:

$$e^{At} = c_0(t)I + c_1(t)A + c_2(t)A^2 + ... + c_{n-2}(t)A^{n-2} + c_{n-1}(t)A^{n-1} \quad (6.5)$$

Because the matrix A satisfies this condition

$$A^n = A^{n-1} + A^{n-2},$$

then we find the characteristic polynomial of the matrix A and determine the eigenvalues of the matrix A , which are three distinct eigenvalues $\lambda_1 = 0$ *and* λ_2, λ_3. The first eigenvalue $\lambda_1 = 0$ of the matrix A is $(n-2)$ eigenvalues. To find the coefficients

$$c_0(t), c_1(t), ..., c_{n-1}(t)$$

by comparing the equations (6.2) and (6.5) we have the matrix equation as the following:

$$(c_0(t) - 1)I + (c_1(t) - \frac{t}{1!})\lambda_i + (c_2(t) - \frac{t^2}{2!})\lambda_i^{\,2} + ... +$$

$$+ (c_{n-2}(t) - \alpha_1(t))\lambda_i^{\,n-2} + (c_{n-1}(t) - \alpha_2(t))\lambda_i^{\,n-1} = 0$$

(6.6)

As we mentioned above the matrix A has three eigenvalues and by using Vandermonde matrix and replace $\lambda_i = s_i$ we construct the matrix equation as the following:

$$\begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^{n-1} \\ 1 & \lambda_3 & \lambda_3^2 & \ldots & \lambda_3^{n-1} \end{bmatrix} \begin{bmatrix} c_0(t)-1 \\ c_1(t)-\frac{t}{1!} \\ \vdots \\ c_{n-2}(t)-\alpha_1(t) \\ c_{n-1}(t)-\alpha_2(t) \end{bmatrix} = 0 \ \ , n > 3 \tag{6.7}$$

By looking to the matrix equation (6.7), there are two or more than two solutions, so we can't obtain the exact form of $\alpha_1(t),\alpha_2(t)$ by using

$$c_0(t),c_1(t),c_2(t),.....,c_{n-2},c_{n-1}(t).$$

To avoid this problem and compute

$$c_0(t),c_1(t),c_2(t),.....,c_{n-2},c_{n-1}(t),$$

we substitute $\lambda_1 = 0$ in (6.6), we get $c_0(t) = 1$, and after the derivative of (6.6) with respect to λ_1 we get

$$(c_1(t)-\frac{t}{1!}) + 2(c_2(t)-\frac{t^2}{2!})\lambda_i^{\,2} + ...+(n-2)(c_{n-2}(t)-$$

$$-\alpha_1(t))\lambda_i^{\,n-3} + (n-1)(c_{n-1}(t)-\alpha_2(t))\lambda_i^{\,n-2} = 0 \tag{6.8}$$

By substituting $\lambda_1 = 0$ in (6.8) we get

$$c_1(t) = \frac{t}{1!},$$

continue this process (n-3) derivation and substitute $\lambda_1 = 0$ to get

$$c_2(t),.....,c_{n-3}(t).$$

Finally, we get

$$(c_{n-2}(t)-\alpha_1(t))\lambda_i^{n-2}+(c_{n-1}(t)-\alpha_2(t))\lambda_i^{n-1}=0 \tag{6.9}$$

Now, we construct the matrix equation as a linear system by taking λ_2,λ_3 in (6.9) we get

$$\begin{bmatrix}\lambda_2^{n-2} & \lambda_2^{n-1}\\ \lambda_3^{n-2} & \lambda_3^{n-1}\end{bmatrix}\begin{bmatrix}c_{n-2}(t)-\alpha_1(t)\\ c_{n-1}(t)-\alpha_2(t)\end{bmatrix}=0 \tag{6.10}$$

Where determinate of (6.10) doesn't equal zero. Then, we conclude that

$$c_{n-2}(t)=\alpha_1(t)\ \ and\ \ c_{n-1}(t)=\alpha_2(t)\ .$$

Lemma (1):

Let $A\in C^{n\times n}$ and satisfies this condition $A^n=A^{n-1}+A^{n-2}$, then the exponential matrix computed by

$$e^{At}=I+tA+\frac{t^2}{2!}A^2+....+c_{n-2}(t)A^{n-2}+c_{n-1}(t)A^{n-1} \tag{6.11}$$

Where $c_{n-2}(t)$ *and* $c_{n-1}(t)$ are computed by

$$\begin{pmatrix}c_{n-2}(t)\\ c_{n-1}(t)\end{pmatrix}=\begin{bmatrix}\lambda_2^{n-2} & \lambda_2^{n-1}\\ \lambda_3^{n-2} & \lambda_3^{n-1}\end{bmatrix}^{-1}\begin{pmatrix}e^{\lambda_2 t}-\sum_{i=0}^{n-3}\frac{t^i\lambda_2^i}{i!}\\ e^{\lambda_3 t}-\sum_{i=0}^{n-3}\frac{t^i\lambda_3^i}{i!}\end{pmatrix}$$

$$and\ \ \lambda_2=\frac{1+\sqrt{5}}{2},\lambda_3=\frac{1-\sqrt{5}}{2} \tag{6.12}$$

Proof:

From equations (6.2) and (6.5) and we can rewrite them and get

$$e^{At} = I + tA + \frac{t^2 A^2}{2!} + ... + \alpha_1(t)A^{n-2} + \alpha_2(t)A^{n-1}$$

$$e^{At} = c_0(t)I + c_1(t)A + c_2(t)A^2 + ... + c_{n-2}(t)A^{n-2} + c_{n-1}(t)A^{n-1}$$

and from equation (6.8) we have from that process

$$c_0(t) = I\ , c_1(t) = \frac{t}{1!}\ , c_2 = \frac{t^2}{2!}.........$$

$$and\ \ c_{n-2}(t) = \alpha_1(t)\ , c_{n-1}(t) = \alpha_2(t)$$

then after comparing them and substitution by

$$c_{n-2}(t) = \alpha_1(t)\ , c_{n-1}(t) = \alpha_2(t)$$

in equation (6.2) we get,

$$e^{At} = I + tA + \frac{t^2 A^2}{2!} + ... + c_{n-2}(t)A^{n-2} + c_{n-1}(t)A^{n-1}$$

Now, replacing A by $\lambda_i, i = 2,3$, we get the exponential matrix as following:

$$e^{\lambda_i t} = I + t\lambda_1 + \frac{t^2 \lambda_i^2}{2!} + ... + c_{n-2}(t)\lambda_i^{n-2} + c_{n-1}(t)\lambda_i^{n-1} \tag{6.13}$$

To compute

$$c_{n-2}(t)\ and\ \ c_{n-1}(t)$$

from equation (6.13) after simplifying it we get,

$$e^{\lambda_i t} - \sum_{i=0}^{n-3} \frac{t^i \lambda_i^i}{i!} = c_{n-2}(t)\lambda_i^{n-2} + c_{n-1}(t)\lambda_i^{n-1} \tag{6.14}$$

Take λ_2 in (6.14) and take λ_3 to construct the matrix equation to compute $c_{n-2}(t)$ and $c_{n-1}(t)$ as following:

$$\begin{pmatrix} c_{n-2}(t) \\ c_{n-1}(t) \end{pmatrix} = \begin{bmatrix} \lambda_2^{n-2} & \lambda_2^{n-1} \\ \lambda_3^{n-2} & \lambda_3^{n-1} \end{bmatrix}^{-1} \begin{pmatrix} e^{\lambda_2 t} - \sum_{i=0}^{n-3} \frac{t^i \lambda_2^i}{i!} \\ e^{\lambda_3 t} - \sum_{i=0}^{n-3} \frac{t^i \lambda_3^i}{i!} \end{pmatrix} . \tag{6.15}$$

Lemma (2):

Let $A \in C^{n \times n}$ and satisfies $A^n = \omega_1 A^{n-1} + \omega_2 A^{n-2}$, then the exponential matrix computed by,

$$e^{At} = I + tA + \frac{t^2}{2!} A^2 + ... + c_{n-2}(t) A^{n-2} + c_{n-1}(t) A^{n-1} \tag{6.16}$$

Where $c_{n-2}(t)$ and $c_{n-1}(t)$ are computed by,

$$\begin{pmatrix} c_{n-2}(t) \\ c_{n-1}(t) \end{pmatrix} = \begin{bmatrix} \lambda_2^{n-2} & \lambda_2^{n-1} \\ \lambda_3^{n-2} & \lambda_3^{n-1} \end{bmatrix}^{-1} \begin{pmatrix} e^{\lambda_2 t} - \sum_{i=0}^{n-3} \frac{t^i \lambda_2^i}{i!} \\ e^{\lambda_3 t} - \sum_{i=0}^{n-3} \frac{t^i \lambda_3^i}{i!} \end{pmatrix} \tag{6.17}$$

now, if ω_1 *and* ω_2 *satisfy this equation* $\omega_1^2 + 4\omega_2 > 0$ *then, we compute*

λ_2 *and* λ_3 *as* $\lambda_2 = (\omega_1 + \sqrt{(\omega_1^2 + 4\omega_2)}\ /2)$ *and* $\lambda_3 = (\omega_1 - \sqrt{(\omega_1^2 + 4\omega_2)}\ /2)$.

now, if ω_1 *and* ω_2 *satisfy this equation* $\omega_1^2 + 4\omega_2 < 0$ *then, we compute* λ_2

and λ_3 *as* $\lambda_2 = (\omega_1 + \sqrt{(-\omega_1^2 - 4\omega_2 i)}\ /2)$ *and* $\lambda_3 = (\omega_1 - \sqrt{(-\omega_1^2 - 4\omega_2 i)}\ /2)$.

Notation (1):

If $\omega_1 = \omega_2$ this mean $\omega_1^2 + 4\omega_2 = 0,$ in this case, we need to compute the $c_{n-2}(t)$ *and* $c_{n-1}(t)$ by derivative,

$$e^{\lambda_2 t} - \sum_{i=0}^{n-3} \frac{t^i \lambda_2^i}{i!} = c_{n-2}(t)\lambda_2^{n-2} + c_{n-1}(t)\lambda_2^{n-1}$$

With respect to λ_2 in Equation (6.8), we get,

$$te^{\lambda_2 t} - \sum_{i=1}^{n-3} \frac{it^i \lambda_2^{i-1}}{i!} = (n-2)c_{n-2}(t)\lambda_2^{n-3} + (n-1)c_{n-1}(t)\lambda_2^{n-2}.$$

Where $c_{n-2}(t)$ *and* $c_{n-1}(t)$ are computed by

$$\begin{pmatrix} c_{n-2}(t) \\ c_{n-1}(t) \end{pmatrix} = \begin{bmatrix} \lambda_2^{n-2} & \lambda_2^{n-1} \\ (n-2)\lambda_2^{n-3} & (n-1)\lambda_2^{n-2} \end{bmatrix}^{-1} \begin{pmatrix} e^{\lambda_2 t} - \sum_{i=0}^{n-3} \frac{t^i \lambda_2^i}{i!} \\ te^{\lambda_2 t} - \sum_{i=1}^{n-3} \frac{it^i \lambda_3^{i-1}}{i!} \end{pmatrix}. \qquad (6.18)$$

Lemma (3):

Let $A \in C^{n\times n}$ and satisfies this condition $A^n = A^{n-1} + A^{n-2} + A^{n-3}$, then the exponential matrix is computed by,

$$e^{At} = I + tA + \frac{t^2}{2!}A^2 + ... + c_{n-3}(t)A^{n-3}c_{n-2}(t)A^{n-2} + c_{n-1}(t)A^{n-1} \qquad (6.19)$$

Where $c_{n-3}(t), c_{n-2}(t)$ *and* $c_{n-1}(t)$ are computed by,

$$\begin{pmatrix} c_{n-3}(t) \\ c_{n-2}(t) \\ c_{n-1}(t) \end{pmatrix} = \begin{bmatrix} \lambda_2^{n-3} & \lambda_2^{n-2} & \lambda_3^{n-1} \\ \lambda_3^{n-3} & \lambda_3^{n-2} & \lambda_3^{n-1} \\ \lambda_4^{n-3} & \lambda_4^{n-2} & \lambda_4^{n-1} \end{bmatrix}^{-1} \begin{pmatrix} e^{\lambda_2 t} - \sum_{i=0}^{n-4} \frac{t^i \lambda_2^i}{i!} \\ e^{\lambda_3 t} - \sum_{i=0}^{n-4} \frac{t^i \lambda_2^i}{i!} \\ e^{\lambda_4 t} - \sum_{i=0}^{n-4} \frac{t^i \lambda_2^i}{i!} \end{pmatrix}$$

$$\text{where } \lambda_2, \lambda_3 \text{ and } \lambda_4 \text{ are root of } \lambda^3 - \lambda^2 - \lambda - 1 = 0 \quad . \qquad (6.20)$$

Lemma (4):

Let $A \in C^{n\times n}$ and satisfies this condition $A^n = \omega_1 A^{n-1} + \omega_2 A^{n-2} + \omega_3 A^{n-3}$,then the exponential matrix is computed by,

$$e^{At} = I + tA + \frac{t^2}{2!}A^2 + ... + c_{n-3}(t)A^{n-3}c_{n-2}(t)A^{n-2} + c_{n-1}(t)A^{n-1} \tag{6.21}$$

Where $c_{n-3}, c_{n-2}(t) \text{ and } c_{n-1}(t)$ are computed by

$$\begin{pmatrix} c_{n-3}(t) \\ c_{n-2}(t) \\ c_{n-1}(t) \end{pmatrix} = \begin{bmatrix} \lambda_2^{n-3} & \lambda_2^{n-2} & \lambda_3^{n-1} \\ \lambda_3^{n-3} & \lambda_3^{n-2} & \lambda_3^{n-1} \\ \lambda_4^{n-3} & \lambda_4^{n-2} & \lambda_4^{n-1} \end{bmatrix}^{-1} \begin{pmatrix} e^{\lambda_2 t} - \sum_{i=0}^{n-4} \frac{t^i \lambda_2^i}{i!} \\ e^{\lambda_3 t} - \sum_{i=0}^{n-4} \frac{t^i \lambda_2^i}{i!} \\ e^{\lambda_4 t} - \sum_{i=0}^{n-4} \frac{t^i \lambda_2^i}{i!} \end{pmatrix} \tag{6.22}$$

where λ_2, λ_3 *and* λ_4 *are the different roots of* $\lambda^3 - \omega_1\lambda^2 - \omega_2\lambda - \omega_3 1 = 0$

Notation (2):

If $\lambda_2, \lambda_3 \text{and } \lambda_4$ are not the different roots, we can also compute the parameters $c_{n-3}, c_{n-2}(t) \text{ and } c_{n-1}(t)$ by using the method as in notation (1).

Lemma (5):

Let $A \in C^{n\times n}$ and satisfies this condition

$$A^n = \omega_1 A^{n-1} + \omega_2 A^{n-2} + \omega_3 A^{n-3} + ... + \omega_k A^{n-k}, k < n,$$

Then the exponential matrix is computed by,

$$e^{At} = I + tA + \frac{t^2}{2!}A^2 + ... + \frac{t^{n-k-1}}{(n-k-1)!}A^{n-k-1} + c_{n-k}(t)A^{n-k} +$$

$$+ c_{n-k+1}(t)A^{n-k+1} + ... + c_{n-1}(t)A^{n-1} \tag{6.23}$$

Where $c_{n-k}(t),.........,c_{n-3}(t),c_{n-2}(t)$ and $c_{n-1}(t)$ are computed by,

$$\begin{bmatrix} c_{n-k}(t) \\ c_{n-k+1}(t) \\ \vdots \\ c_{n-1}(t) \end{bmatrix} = \begin{bmatrix} \lambda_2^{n-k} & \lambda_2^{n-k+1} & \cdots & \lambda_2^{n-1} \\ \lambda_3^{n-k} & \lambda_3^{n-k+1} & \cdots & \lambda_3^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{k+1}^{n-k+1} & \lambda_{k+1}^{n-k+1} & \cdots & \lambda_{k+1}^{n-1} \end{bmatrix}^{-1} \begin{bmatrix} e^{\lambda_2 t} - \sum_{i=0}^{n-k-1} \frac{t^i \lambda_2^i}{i!} \\ e^{\lambda_3 t} - \sum_{i=0}^{n-k-1} \frac{t^i \lambda_3^i}{i!} \\ \vdots \\ e^{\lambda_{k+1} t} - \sum_{i=0}^{n-k-1} \frac{t^i \lambda_{k+1}^i}{i!} \end{bmatrix} \qquad (6.24)$$

$\lambda_2, \lambda_3, .., \lambda_{k+1}$ *are the different roots of* $\lambda^k - \omega_1 \lambda^{k-1} - ... - \omega_{k-1}\lambda - \omega_k 1 = 0$.

6.3-Applications of method

Here, we will give some example to show the procedure and effectiveness this method.

Example (1):

Let A a matrix (4×4), $A = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}$

After some calculations we get $A^4 = A^3 + 2A^2$, then, $\omega_1 = 1$ *and* $\omega_2 = 2$. From lemma (2), we have ω_1 *and* ω_2 satisfy $\omega_1 + 4\omega_2^2 > 0$ .then, we compute λ_1 *and* λ_2 as in lemma (2) which equals to $\omega_1 = 2$ *and* $\omega_2 = -1$. Then, the exponential matrix will be as the following:

$$e^{At} = I_4 + tA + c_2(t)A^2 + c_3(t)A^3$$

$$= I_4 + tA + (\frac{1}{12}e^{2t} + \frac{2}{3}e^{-t} + \frac{1}{2}t - \frac{3}{4})A^2 + (\frac{1}{12}e^{2t} - \frac{1}{3}e^{-t} - \frac{1}{2}t + \frac{1}{4})A^3$$

Where

$$c_2 = (\frac{1}{12}e^{2t} + \frac{2}{3}e^{-t} + \frac{1}{2}t - \frac{3}{4}), c_3(t) = (\frac{1}{12}e^{2t} - \frac{1}{3}e^{-t} - \frac{1}{2}t + \frac{1}{4})$$

then, the exponential matrix is

$$e^{At} = \begin{bmatrix} 1 & 2t & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{2t} & 0 \\ -e^{-t} & 2e^{-t} + 2t - 2 & 0 & e^{-t} \end{bmatrix}$$

Example (2):

Let A a matrix (4×4), $A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & -1 \end{bmatrix}$

As is the above example, the matrix A satisfy $A^4 = 2A^3 + A^2 - 2A$, then, $\omega_1 = 2, \omega_2 = 1$ and $\omega_3 = -2$ from lemma (4), we compute λ_2, λ_3 and λ_4 then, the exponential matrix will be as the following:

$$e^{At} = I_4 + c_1(t)A + c_2(t)A^2 + c_3(t)A^3$$

$$= I_4 + (\frac{-1}{6}e^{2t} + e^t - \frac{1}{3}e^{-t} - \frac{1}{2})A + (\frac{1}{2}e^t + \frac{1}{2}e^{-t} - 1)A^2 +$$

$$+ (\frac{1}{6}e^{2t} - \frac{1}{2}e^t - \frac{1}{6}e^{-t} + \frac{1}{2})A^3$$

Where

$$c_1 = (\frac{-1}{6}e^{2t} + e^t - \frac{1}{3}e^{-t} - \frac{1}{2}), c_2 = (\frac{1}{2}e^t + \frac{1}{2}e^{-t} - 1)$$

$$and \ \ c_3(t) = (\frac{1}{6}e^{2t} - \frac{1}{2}e^t - \frac{1}{6}e^{-t} + \frac{1}{2})$$

then the exponential matrix is

$$e^{At} = \begin{bmatrix} e^{2t} & 0 & 0 & 0 \\ 0 & e^t & e^t - 1 & e^t + e^{-t} - 2 \\ 0 & 0 & 1 & -2e^{-t} + 2 \\ 0 & 0 & 0 & e^{-t} \end{bmatrix}$$

6.4- Conclusions

In this chapter we presented new method to compute the exponential matrix e^{At} as accurate solution. The basic idea of this method is based on the matrix theory; the matrices satisfy the special case

$$A^n = \omega_1 A^{n-1} + \omega_2 A^{n-2} + \omega_3 A^{n-3} + ... + \omega_k A^{n-k}, k < n.$$

Furthermore, this method can be extended to the more general case

$$A^n = \omega_1 A^{k-1} + \omega_2 A^{k-2} + \omega_3 A^{k-3} + ... + \omega_k A^{k-m}, k < n, m < k.$$