CHAPTER[4] 4: NEW METHOD TO COMPUTE COMMUTING AND NON COMMUTING EXPONENTIAL MATRIX

4.1-Introduction

The matrix exponential is a very important particular matrix functions that has been studied extensively in the last 50 years. In this chapter, we introduce a new method to compute the matrix exponential where this matrix is located in $M(2,R)$. It is known that in the numerical fields for the exponential function $\exp(x) = e^x$ satisfies the equation of exponential function $e^{x+y} = e^x e^y$. This equality is not true in general cases when the exponential function is defined on the matrices especially when non-commutative matrices are used. But it is known that the equation is verified if the two exponential matrices are commutative:

$$AB = BA \quad then \quad e^{A+B} = e^A e^B = e^B e^A$$

But when we applied the converse for this expression we find that it is not always true for two exponential commutative matrices, but we can apply any of the following relations:

$$e^{A+B} = e^A e^B = e^B e^A \ , \ \ e^{A+B} \neq e^A e^B = e^B e^A$$

$$e^{A+B} = e^A e^B \neq e^B e^A \ , \ \ e^{A+B} \neq e^A e^B \neq e^B e^A$$

Several studies have tried to determine the characteristics of the matrices that do not commute in the exponential matrices. In particular, the

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problem has been studied for 50 years for matrices which has two or three dimension for more details see [1,10,15, 18, 21], and this problem has been discussed recently in [14].The simplest case is that of the matrices in $M(2,R)$, discussed and solved in [14] in the more general of complex algebras of degree two.

This chapter proposes a simple discussion on how to characterize the matrices of $M(2,R)$ for which we have:

$$e^A e^B = e^B e^A = e^{A+B} \tag{4.1}$$

and it shows that there are not matrices, for which we have:

$$e^A e^B = e^B e^A \neq e^{A+B}.$$

4.2- Definitions & terminology

The set of real matrices $2 \times 2, M(2,R)$ is a vector space over R with respect to the operations of the matrix addition and multiplication by a real number, in algebra is not commutative with respect to those operations and the usual matrix product, but it is a complete metric space compared to the usual norm

$$\|A\| = \sup_{|x|=1} |Ax| \ .$$

In the following we will consider matrices in $M(2,R)$, such that a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible (non-singular) if and only if the (det $A = ad - bc \neq 0$) and its inverse is given by

$$A^{-1} = \frac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

The set of invertible matrices, denoted by $GL(2,R)$ is a non-commutative group under the operation of the matrices product, whose identity element is the matrix:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The trace of a matrix is the sum of its elements on the main diagonal:

$$trace\, A = trace \begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + d$$

The commutator of two matrices A, B is the matrix which is defined by:

$$[A,B] = AB - BA$$

If $[A,B] \neq 0$ then A, B and I are linearly independent. The centre

$$C(2,R) \text{ of } M(2,R)$$

is the set of matrices $X \in M(2,R)$ that commute with all matrices in $M(2,R)$:

$$C(2,R) = \{X : [A,X] = 0, \forall A \in M(2,R)\}$$

and is the subgroup of $GL(2,R)$ constituted by the scalar matrices:

$$X = xI \;\; with \;\; x \in R. \;\; C(2,R)$$

is a Lie group and is obviously isomorphic to R^*. The sign of

$X \in C(2,R)$ is the sign of $x \in R$.

4.3-Exponential Matrix

The exponential matrix is defined by:

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!} = I + A + \frac{A^2}{2!} + \ldots\ldots + \frac{A^{n-1}}{(n-1)!} + \ldots\ldots \qquad (4.2)$$

The series (4.2) is absolutely convergent and defines an entire function in C, so it is convergent in the metric space $M(2,R)$ because the product of matrices in $M(2,R)$ is not commutative. Although its exponential function is well-defined, it does not satisfy the equation (4.1) in general. For more details and examples of these properties see [1,10,33].

4.4-Main Results for computing the exponential matrix

Computing of the exponential matrix $M(n,K)$ quickly becomes very complicated as n increases. However, there are procedures that allow always make such a calculation in a finite number of steps, for more information about these methods see [17,19,20,21,22]. In the case of matrices $M(2,R)$ the calculation of the exponential matrix is quite simple with the following decomposition.

Lemma (4.1):

Each matrix $A \in M(2,R)$ can be decomposed into a sum of two matrices one of them is in the center $C(2,R)$ and the other has null trace:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = kI + A' = k\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} m & b \\ c & -m \end{pmatrix}$$

And, for two matrices $A, B \in M(2, R)$, we have:

$$[A', B'] = 0 \Leftrightarrow [A, B] = 0.$$

Proof:

The decomposition is obvious, just put:

$$k = \frac{a+d}{2} = \frac{trace\ A}{2},\ m = \frac{a - d}{2}$$

then, $A' = A - \frac{trace\ A}{2} I$

Then, we have

$$[A', B'] = \left[A - \frac{trace\ A}{2} I, B - \frac{trace\ B}{2} I \right] =$$

$$= [A, B] - \left[A, \frac{trace\ B}{2} I \right] - \left[\frac{trace\ A}{2} I, B \right] +$$

$$+ \left[\frac{trace\ A}{2} I, \frac{trace\ B}{2} I \right] = [A, B]$$

Since, we conclude that $\frac{trace\ A}{2} I, \frac{trace\ B}{2} I \in C(2, R).$

For a traceless matrix, the series which defines the exponential function is particularly easy to calculate, as shown by the following lemma.

Lemma (4.2):

If M is a traceless matrix then:

$$e^{M} = I\cos\theta + M\frac{\sin\theta}{\theta}, \text{ where } \theta = \sqrt{\det M}.$$

Proof:

Firstly, we start by noting that for a traceless matrix:

$$M = \begin{pmatrix} m & b \\ c & -m \end{pmatrix}$$

We have,

$$M^{2} = \begin{pmatrix} m & b \\ c & -m \end{pmatrix}\begin{pmatrix} m & b \\ c & -m \end{pmatrix}$$

$$= \begin{pmatrix} m^{2} + bc & o \\ 0 & bc + m^{2} \end{pmatrix} = -\det(M)I$$

then, we put $\theta = \sqrt{\det M}$ we get

$$e^{M} = \sum_{k=0}^{\infty}\frac{M^{k}}{k!} = I + M - \frac{\theta^{2} I}{2!} - \frac{\theta^{3} M}{3!\,\theta} + \frac{\theta^{4} I}{4!} + \frac{\theta^{5} M}{5!\,\theta} - \frac{\theta^{6} I}{6!} + \dots\dots$$

$$= I(1 - \frac{\theta^{2}}{2!} + \frac{\theta^{4}}{4!} \dots\dots\dots) + \frac{M}{\theta}(\frac{\theta}{1!} - \frac{\theta^{3}}{3!} + \frac{\theta^{5}}{5!} \dots\dots\dots)$$

$$= I\cos\theta + \frac{M}{\theta}\sin\theta.$$

We can therefore calculate the exponential matrix $M(2,R)$ by using the following theorem.

Theorem (4.3):

For each matrix $A \in M(2,R)$, we have:

$$e^{A} = e^{kI + A'} = e^{kI} e^{A'} = e^{k} \left(I \cos\alpha + A' \frac{\sin\alpha}{\alpha} \right)$$

With, $$k = \frac{trace\,A}{2}, A' = A - kI,\ \alpha = \sqrt{\det A'}$$

Proof:

We use the decomposition of Lemma (4.1) and the commutator by noting that $[kI, A'] = 0$, we have:

$$e^{A} = e^{kI + A'} = e^{kI} e^{A'} = e^{A'} e^{kI}$$

and by Lemma (4.2) we obtain the desired .

Note: Note that the result of this theorem is valid even if $\det A' \le 0$.

If

$$\det A' = 0 \text{, then } \alpha = 0,$$

Just put $\frac{\sin 0}{0} = 1$ and in this case we have:

$$e^{A} = e^{k}(I + A').$$

If

$$det\,A' < 0\text{, then } \alpha = \pm i|\alpha| = \pm i\sqrt{|det\,A'|},$$

it takes, as in the real case, the root with + sign and use the relationships between rotation functions and hyperbolic functions

$$cosh(\alpha) = cos(i\alpha) \qquad cos(\alpha) = cosh(i\alpha)$$

$$sinh(\alpha) = isin(i\alpha) \qquad sin(\alpha) = isinh(i\alpha)$$

to obtain:

$$\frac{sin(i|\alpha|)}{i|\alpha|} = \frac{i\,sinh|\alpha|}{i|\alpha|} = \frac{e^{|\alpha|} - e^{-|\alpha|}}{2|\alpha|},$$

$$cos(i|\alpha|) = cosh|\alpha| = \frac{e^{|\alpha|} + e^{-|\alpha|}}{2}$$

It proves easily that if $A \in C(2,R)$ then $e^A \in C(2,R)$:

$$A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \Rightarrow e^A = \begin{pmatrix} e^a & 0 \\ 0 & e^a \end{pmatrix}$$

but the contrary implication is not generally true, as can be seen from the following lemma.

Lemma (4.4):

Let A a nonzero matrix, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *then* $e^A \in C(2,R)$ if and only if

$$A \in C(2,R)\ or\ \ det\,A' = det\,A - (\frac{trace\,A}{2})^2 = \mu^2\pi^2\ with\ \mu \in N^+ \ .$$

Proof:

If $e^A \in C(2,R)$, it must be:

$$e^A = e^k(I\cos\alpha + A'\frac{\sin\alpha}{\alpha})$$

$$= e^k\begin{pmatrix} \cos\alpha + \frac{m\sin\alpha}{\alpha} & \frac{b\sin\alpha}{\alpha} \\ \frac{c\sin\alpha}{\alpha} & \cos\alpha - \frac{m\sin\alpha}{\alpha} \end{pmatrix}$$

by equating the terms on the secondary diagonal we have:

$$\frac{b\sin\alpha}{\alpha} = 0 \Rightarrow b = 0 \;\; or \;\; \frac{\sin\alpha}{\alpha} = 0$$

$$\frac{c\sin\alpha}{\alpha} = 0 \Rightarrow c = 0 \;\; or \;\; \frac{\sin\alpha}{\alpha} = 0$$

And the subtracting the terms on the main diagonal, we obtain:

$$\frac{2m\sin\alpha}{\alpha} = 0 \Rightarrow m = 0 \;\; or \;\; \frac{\sin\alpha}{\alpha} = 0$$

Now if

$$b = c = m = 0$$

the matrix A' is zero, and then, $A \in C(2,R)$. Otherwise we must have

$$\frac{\sin\alpha}{\alpha} = 0$$

And there are three possible cases as following:

$$det\,A' = 0 \Rightarrow \frac{sin\alpha}{\alpha} = 1$$

$$deA' > 0 \Rightarrow \frac{sin\alpha}{\alpha} = 0 \Rightarrow \alpha = \mu\pi \;\; with\; \mu \in N^{+}$$

$$det\,A' < 0 \Rightarrow \frac{sin\alpha}{\alpha} = \frac{sin\,i|\alpha|}{i|\alpha|} = \frac{sinh|\alpha|}{|\alpha|} > 0 \quad \forall|\alpha| > 0$$

and since $det\,A' = \alpha^2$ has this structure.

4.5-Conclusion

There are many ways to compute the exponential matrix. The current study provides a new method to compute the exponential matrix in which this matrix is located in $M(2,R)$. The study also presents some lemmas and theorems that explain the procedure of this method.