Chapter 1

Introduction

1.1. Introduction

Before going into the subject matter of this thesis we want to draw up some attention to the history of vector integration which seems necessary to be familiar with this work. For this purpose a brief but systematic description of the development of vector integration is given here. There are three types of generalizations of the classical Lebesgue integration, namely, the integrand vector-valued and the measure scalar-valued, the integrand scalar-valued and the measure vector-valued and both the integrand and the measure vector-valued. In this thesis we are mainly interested in the integration theory of vector-valued functions with respect to vector-valued measures.

The most successful version of integrals for vector-valued functions and scalar-valued measures defined on $\sigma$-algebras were introduced by Bochner [2] in 1933 which is known as Bochner integral. He observed that most of the usual theorems connected with the Lebesgue integral remained valid with respect to the $L_1$-norm which was being obtained replacing absolute value by norm.

Let $(\Omega, \Sigma, \mu)$ be a finite measure space and $X$ be a real Banach space. Let $L_1(\mu, X)$ denote the space of all Bochner integrable functions $f : \Omega \to X$. It is well-known that $L_1(\mu, X)$ is a Banach space with respect to the norm

$$\|f\|_1 = \int_{\Omega} \|f\| \, d\mu.$$ 

The Banach space $L_1(\mu, X)$ has been extensively studied by numerous authors and many of its Banach space properties are now well known in the literature, such as Dunford and Schwartz [25], Dinculeanu [14], Diestel and Uhl. [16], Cartwright [4], Diestel [15] and Talagrand [67], to mention a few.
In his famous paper [51], Pettis proved that a Banach space valued function is measurable if and only if it is scalarly measurable and essentially separably valued. A theory of integration similar to the Bochner integration is impossible for functions that are only scalarly measurable and as such several mathematicians began to search for an integration theory for the scalarly measurable functions and as a consequence, the ideas of Dunford integral, Pettis integral and Gelfand integral emerged.

The integral that we call the ‘Dunford integral’ traces its life back to [24]. Gelfand and Dunford proved that every scalarly integrable function with values in a Banach space is always Dunford integrable. Also it was shown by a counter example that the Dunford integral is not countably additive and so not absolutely continuous.

The definition of Pettis integral over a real interval was suggested by Gelfand [31] and extended by Pettis [51] to the case of a finite abstract measure space. Pettis also discussed the basic properties of this integral and its relation to the other integrals such as Bochner, Birkhoff and Gelfand. No one else was able to unravel additional informations about its structure. The Pettis integral was studied by Dunford [23] and correctly should be called Dunford’s second integral.

Gelfand integral deals with a function which takes its values in the dual of a Banach space. Gelfand proved that every weak* scalarly integrable function is always Gelfand integrable and that the Dunford integral can be thought of as a particular case of Gelfand integral.

After the idea of Pettis integration being introduced in the thirties of the last century, there had been no significant progress in its study. Even in 1977, the view about Pettis integration was ‘Presently the Pettis integral has very few applications. But our prediction is that when (and if ) the general Pettis integral was understood it will pay off in deep applications’ [16, p.57]. The prediction has been proved to be true as recorded in the two seminal papers of Edgar [26], [27] and also the survey of Diestel and Uhl [17] and the memoir of Talagrand [66]. It has further been developed
by the works of Huff [34], Musial [43], Geitz [29, 30] and Stefánsson [57, 59, 60], among others.

Let $P_1(\mu, X)$ denote the space of all scalarly measurable Pettis integrable functions. It has been shown by Pettis in his paper [51] that $P_1(\mu, X)$ is a normed linear space, endowed with the norm given by

$$\|f\|_p = \sup \left\{ \left| \int \omega x^* f \, d\mu \right| : x^* \in B_{X^*} \right\}.$$

He also showed by a counter example that $P_1(\mu, X)$ may not be complete. After a long gap of five decades it was shown by Drewnowski et al. [22] that $P_1(\mu, X)$ is barrelled.

In 1967, Dinculeanu [14] extensively studied the integration of vector-valued functions with respect to a vector measure. In fact, most of the works on vector measure and integration that had come in the literature from 1950 to 1966 has been included in his book [14] rigorously. Since then significant work on vector integration started rapidly.

I. Dobrakov developed a theory of integration for vector-valued functions with respect to operator-valued measures defined on $\delta$-rings in a series of papers initiated by his fundamental papers in 1970 in [18, 19].

In 2004, Dobrakov and Panchapagesan [21] provided detailed proofs of many results of [18] and [19] and discussed some of the distinguishing features of this theory including the stronger version of the Pettis measurability criteria.

In 1980, Swartz [63] gave an alternative definition for the Dobrakov integral which is a 'weak type' integral in the spirit of the Pettis integral. He showed that a function is Dobrakov integrable if and only if it is integrable in the sense as defined by him in [63].
In [50] Barrière also studied a theory of integration based on the notion of semivariation and developed the theories of bilinear integration and tensor integration and proved Dominated Convergence Theorem and Convergence Theorem of Vitali type from this notion.

In [54], Rodriguez studied the integration theory of vector-valued functions with respect to operator-valued measures and extended the theories of Birkhoff and Macshane integrals and connected these integrals with the \( S^* \)-integrals as developed by Dobrakov in [20].

The integration theory of scalar-valued functions with respect to a vector-valued measure defined on \( \sigma \)-algebra was introduced by Bartle, Dunford and Schwartz in 1955 [1]. Later, in 1970, Lewis [37] constructed an integration theory of scalar-valued functions with respect to a vector-valued measure in a locally convex Hausdorff topological vector space (in brief, locally convex space), where the value of the integral is defined by duality. In the case of a Banach space setting, this theory is equivalent to the theory of Bartle-Dunford-Schwartz as mentioned above.

Let \( L_1(\nu) \) denote the space of all scalar-valued integrable functions \( f : \Omega \to \mathbb{R} \) with respect to the vector measure \( \nu : \Sigma \to X \) where \( X \) is a Banach space. The space \( L_1(\nu) \) has been thoroughly investigated and many of its properties up to 1975 can be found in the book of Kluvánek and Knowles [36]. It was shown by Kluvánek and Knowles [36, p.73 and p.78] and by Swartz [62] in 1976 that \( L_1(\nu) \) is a Banach space with respect to the norm

\[
\| f \|_1 = \sup \left\{ \int_A | f | \, d \nu : x^* \in B_{x^*} \right\}.
\]

The space \( L_1(\nu) \) has been further studied by various authors in their works. We mention a few of these works in the following:

Ricker [52], [53]; Okada [44]; Curbera [8], [9], [10]; Okada and Ricker [45], [46], [47]; Okada et al. [48]; Sánchez-Pérez [55], [56] and Stefánsson [58].
Vector measures defined on $\sigma$-algebras have become a very important tool for the study of operators $T : Z \to Y$ between Banach functions spaces. In fact, the optimal domain $T$ can be described as the space $L_1(\nu)$ of integrable functions with respect to the vector measure $\nu$ canonically associated to $T$ by $\nu(A) = T(\chi_A)$ (see [11, p.133] and [49, Chapter 3 and 4]).

The integration theory of scalar-valued functions with respect to vector measures defined on $\delta$-rings was first introduced by Lewis in 1972 in [38] in a locally convex space setting, thereby extending his results obtained in [37]. He also introduced the idea of scalarly integrable (weakly integrable) functions with respect to a vector measure $\nu$ and established the relationship between scalarly $\nu$-integrable functions and $\nu$-integrable functions in a quasi complete locally convex space $X$. In fact, in a Banach space setting it follows that a scalarly $\nu$-integrable function is $\nu$-integrable if and only if $X$ contains no copy of $c_0$.

Let $w-L_1(\nu)$ denote the space of all scalarly integrable functions $f : \Omega \to \mathbb{R}$ with respect to the vector measure $\nu : \Sigma \to X$ where $X$ is a Banach space. It has been shown by Stefañsson [58] that $w-L_1(\nu)$ is a Banach space containing $L_1(\nu)$ as a closed subspace with respect to the norm

$$
\|f\|_w = \sup \left\{ \int_{\Omega} |d| x^* \nu| : x^* \in B_{x^*} \right\}.
$$

In [8], Curbera showed that $L_1(\nu)$ is an order continuous Banach lattice with weak order unit and is weakly sequentially complete if $X$ contains no copy of $c_0$. Stefañsson [58] also showed that the space $w-L_1(\nu)$ is a $\sigma$-complete Banach lattice which may not be order continuous. He then proved some equivalent conditions for which $L_1(\nu) = w-L_1(\nu)$.

The study of integration theory of scalar-valued functions with respect to the vector measures defined on $\delta$-rings as introduced by Lewis [38] was further continued by Masani and Niemi [41], [42] in 1989. In 2005, Delgado [13] further
developed this theory and analysed the subtle differences between the $L_1$-spaces of vector measures defined on $\delta$-rings and defined on $\sigma$-algebras. In fact, she showed that the space $L_1(\nu)$ of a vector measure $\nu$ defined on a $\delta$-ring is an order continuous Banach lattice which may not have a weak order unit. Since a countably additive vector measure defined on a $\delta$-ring may not be strongly additive, she studied the effect of strong additivity of $\nu$ on $L_1(\nu)$ and connected the analytic properties of $\nu$ with the lattice properties of $L_1(\nu)$.

The theory of $L_p$-spaces, $1 < p < \infty$, of scalar-valued functions with respect to the vector measures $\nu : \Sigma \to X$ was systematically studied and developed by Sánchez-Pérez in [55]. As in the case of $L_1(\nu)$ he showed that $L_p(\nu)$ is an order continuous Banach lattice with weak order unit.

If $(\Omega, \Sigma, \mu)$ is a finite positive measure space and $1 \leq p < \infty$, it is well known that the dual of $L_p(\mu)$ is $L_q(\mu)$ where $\frac{1}{p} + \frac{1}{q} = 1$. But in the case of $L_p$-spaces associated to the vector measure $\nu : \Sigma \to X$, there is not a good representation of the dual of $L_p(\nu)$. In order to overcome this difficulty, Sánchez-Pérez [56] introduced a new concept known as ‘vector measure duality’ and he showed that $$(L_p(\nu))' = L_q(\nu)$$ where $1 < p < \infty$.

In [28] Fernández et al. further studied the space $L_p(\nu)$ and $w\cdot L_p(\nu)$ for $1 < p < \infty$ and showed some equivalent conditions for the equality of the spaces $L_p(\nu)$ and $w\cdot L_p(\nu)$.

In 2001, G.F. Stefánsson [61] defined the tensor integration theory of vector-valued functions $f : \Omega \to X$ with respect to a countably additive vector measure $\nu : \Sigma \to Y$ where the value of the integral is an element of the injective tensor product $X \bar{\otimes} Y$. He investigated many properties of the tensor integral including the Dominated Convergence Theorem. If $L_1(\nu, X, Y)$ denotes the space of all $\bar{\otimes}$-
integrable functions then he showed that $L_1(\nu, X, Y)$ is a Banach space, equipped with the norm

$$
N(f) = \sup \left\{ \int_\Omega \| f \| \, d \| y^* \nu \| : y^* \in B_{Y^*} \right\}.
$$

He also defined the integral of weakly $\nu$-measurable functions and introduced the idea of weakly $\mathcal{G}$-integrable functions and discussed many of its properties.

On the other hand, Jefferies and Okada studied the theory of tensor integration of vector-valued functions with respect to the vector-valued measures defined on a $\sigma$-algebra \[35, Definition 1.5, p. 521\]. Their definition of tensor integration is weaker than the definition as given by Stefañsson in \[61, Definition 1, p. 927\]. However, they succeeded in developing a relationship between the tensor integrable functions and Dobrakov integrable functions \[35, Theorem 3.5 and Corollary 3.6\].

As mentioned earlier, the object of this thesis is to study the integration theory of vector-valued functions with respect to a countably additive vector measures defined on a $\sigma$-algebra or on a $\delta$-ring.

The main object of chapter 2 of the thesis is to study some general properties of the Banach space $L_1(\nu, X, Y)$ such as order continuity, weakly compactly generated, weak sequential completeness and separability. Our results are generalizations of these properties which have already been studied for the spaces $L_1(\mu, X)$ and $L_1(\nu)$ in \[4, 15, 67, 25, 8, 53\] respectively. We have also studied the space $w-L_1(\nu, X, Y)$ of all weakly $\mathcal{G}$-integrable functions and have shown that $w-L_1(\nu, X, Y)$ is a normed linear space with respect to the semi variation norm

$$
\|f\|_w = \sup \left\{ \int_\Omega \| x^* f \| \, d \| y^* \nu \| : x^* \in B_{X^*}, y^* \in B_{Y^*} \right\},
$$

which is not complete. However, we have succeeded in proving that $w-L_1(\nu, X, Y)$ is barrelled, when $\nu$ is non-atomic which is a generalization of the results published in \[22\]. The
Chapter 3 is concerned with the spaces of $p$-tensor integrable functions and related Banach space properties. In this chapter, we have extended the definitions of $L_{1}(\nu, X, Y)$ and $w-L_{1}(\nu, X, Y)$, which have been introduced by Stefánsson in [61] to $L_{p}(\nu, X, Y)$ and $w-L_{p}(\nu, X, Y)$ respectively, where $1 < p < \infty$ and have studied some basic properties of these spaces. We have also studied vector measure duality in $L_{p}(\nu, X, Y)$ for $1 < p < \infty$, which is a generalization of the idea of vector measure duality in $L_{p}(\nu)$ as introduced by Sánchez-Pérez in [56]. The contents of this chapter have been published in Real Analysis Exchange, 34(1), 2008/2009, 87-104 (jointly with N.D.Chakraborty) [6].

The last chapter, namely chapter 4, deals with the integration theory of vector-valued functions with respect to vector measures defined on $\delta$-rings. This extends the theory of scalar-valued integrable functions with respect to vector measures defined on $\delta$-rings as developed in [38], [41], [42] and [13] to the case of vector-valued tensor integrable functions with respect to vector measures defined on $\delta$-rings. In this setting we have proved the Dominated Convergence Theorem for tensor integrable functions and have introduced the idea of weakly Bochner integrable functions and have shown that $w-L_{1}(\nu, X)$, the space of all weakly Bochner integrable functions, is a Banach space with respect to the norm

$$N(f) = \sup_{\delta} \left\{ \int |f| \, d|\nu| : y^* \in B_{y^*} \right\}$$

and an alternative proof of the completeness of $L_{1}(\nu, X, Y)$ as given in [61] follows as a corollary of the completeness of $w-L_{1}(\nu, X)$, when $\nu$ is defined on a $\delta$-ring. The contents of this chapter are scheduled to be published soon in the Illinois Journal of Mathematics (jointly with N.D.Chakraborty) [7].
The references of the books and journals have been presented at the end of the thesis and the corresponding serial numbers have been mentioned in the thesis where those references are needed.

1.2. Notations, Definitions and Preliminaries

In this section, we introduce some notations and recall some definitions and results that will be used throughout this thesis. For this, we closely follow the books of [25], [14], [16] and [40].

All Banach spaces considered in this thesis are over the field of real numbers \( \mathbb{R} \). Throughout this thesis, \( X \) and \( Y \) are two Banach spaces with topological duals \( X' \) and \( Y' \) respectively. \( B_X \) (resp. \( B_{X^*} \)) denotes the closed unit ball of \( X \) (resp. \( X' \)) and \( L(X,Y) \) denotes the Banach space of all bounded linear operators from \( X \) to \( Y \).

1.2.1. Tensor Products

Let us note that \( X \) and \( Y \) are two linear spaces. A mapping \( A \) from the cartesian product \( X \times Y \) to \( \mathbb{R} \) is called a bilinear functional if it satisfies the following conditions:

(i) \( A(\alpha_1 x_1 + \alpha_2 x_2, y) = \alpha_1 A(x_1, y) + \alpha_2 A(x_2, y) \)

and (ii) \( A(x, \beta_1 y_1 + \beta_2 y_2) = \beta_1 A(x, y_1) + \beta_2 A(x, y_2) \)

for all \( x, x \in X, y, y \in Y \) and all scalars \( \alpha_i, \beta_i, i = 1, 2 \).

Let \( B(X \times Y) \) denote the vector space of all bilinear functionals on \( X \times Y \). Now the algebraic tensor product of \( X \) and \( Y \), denoted by \( X \otimes Y \), is a space of all linear functionals on \( B(X \times Y) \), which is defined by

\[ (x \otimes y)(A) = A(x, y), \quad \text{for all } A \in B(X \times Y) \text{ and } x \in X, y \in Y. \]

So \( X \otimes Y \) is a subspace of the dual of \( B(X \times Y) \) spanned by the elements of the form \( x \otimes y, x \in X, y \in Y \).
Thus, a typical tensor in $X \otimes Y$ has the form
\[ u = \sum_{i=1}^{n} \lambda_i \ x_i \otimes y_i , \]
where $x_i \in X, y_i \in Y$, and $\lambda_i$'s are scalars.

Since it can be easily seen that
\begin{align*}
(i) \quad & (x_1 + x_2) \otimes y = x_1 \otimes y + x_2 \otimes y \\
(ii) \quad & x \otimes (y_1 + y_2) = x \otimes y_1 + x \otimes y_2 \\
(iii) \quad & \lambda (x \otimes y) = (\lambda x) \otimes y = x \otimes (\lambda y) \\
(iv) \quad & 0 \otimes y = x \otimes 0 = 0 ,
\end{align*}
by applying the result (iii) above we can rewrite the representation of $u$ as
\[ u = \sum_{i=1}^{n} x_i \otimes y_i . \]

For any $u \in X \otimes Y$, we define $\lambda (u)$ by
\[ \lambda (u) = \sup \{ |(x^* \otimes y^*) (u) | : x^* \in B_{X^*} , \ y^* \in B_{Y^*} \} . \]
Then it is easily verified that $\lambda$ is a norm on $X \otimes Y$. Usually the algebraic tensor product $X \otimes Y$, equipped with the norm $\lambda$, is incomplete. Its completion is denoted by $X \widehat{\otimes} Y$ and is called the injective tensor product of $X$ and $Y$ [16, p.223 and p.225].

The injective tensor product $X \widehat{\otimes} Y$ may be considered as a subspace of $\mathfrak{B}(X^* \times Y^*)$, of all bounded bilinear functionals on $X^* \times Y^*$, or of either of the spaces $L(X^*, Y)$ or $L(Y^*, X)$.

### 1.2.2. Banach lattices

Following Lindenstrauss and Tzafriri [40, Chapter 1], a partially ordered Banach space $X$ is called a Banach lattice if the following conditions are satisfied:
\begin{itemize}
  \item[(i)] $x \leq y$ implies $x + z \leq y + z$, for every $x, y, z \in X$.
  \item[(ii)] $ax \geq 0$, for every $x \geq 0$ in $X$ and every non-negative real $a$.
  \item[(iii)] for all $x, y \in X$ there exists a least upper bound (l.u.b.) $x \vee y$ and a greatest lower bound (g.l.b.) $x \wedge y$.
\end{itemize}
(iv) if \( x, y \in X \) with \( |x| \leq |y| \), then \( \| x \| \leq \| y \| \), where the absolute value \( |x| \) of \( x \) is defined by \( |x| = x \vee (-x) \).

By a sublattice of a Banach lattice \( X \) we mean a closed subspace \( Y \) of \( X \) such that \( x \vee y \) (and so \( x \wedge y = x + y - x \vee y \)) belongs to \( Y \) whenever \( x, y \in Y \).

An ideal in \( X \) is a closed subspace \( Y \) of \( X \) for which \( y \in Y \) whenever \( |y| \leq |x| \) for some \( x \in Y \).

The dual \( X^* \) of a Banach lattice \( X \) is also a Banach lattice provided that its positive cone is defined by \( x^* \geq 0 \) in \( X^* \) if and only if \( x^* (x) \geq 0 \), for every \( x \geq 0 \) in \( X \).

A weak order unit of \( X \) is an element \( e \ (e \geq 0) \in X \) with the property that for \( x \in X \), \( \inf \{ x, e \} = 0 \) implies \( x = 0 \) [40, p.9].

A Banach lattice \( X \) is said to be order complete (\( \sigma \)-order complete) if every order bounded set (sequence) in \( X \) has a least upper bound (l.u.b.).

A subset \( A \) of a Banach lattice \( X \) is called downward directed if for all \( x, y \in A \) there exists \( z \in A \) such that \( z \leq x \) and \( z \leq y \).

A Banach lattice \( X \) is said to have an order continuous norm (\( \sigma \)-order continuous norm) or, briefly, to be order continuous (\( \sigma \)-order continuous) if, for every downward directed set (sequence) \( \{ x_a \}_{a \in A} \) in \( A \subset X \) with \( \wedge_{a \in A} x_a = 0 \), we have \( \lim_{a} \| x_a \| = 0 \).

Let \( X \) be a Banach lattice. Then the following statements are equivalent:

(i) \( X \) is \( \sigma \)-complete and \( \sigma \)-order continuous.

(ii) Every order bounded increasing sequence in \( X \) converges in the norm topology of \( X \).
(iii) \( X \) is order continuous.
(iv) \( X \) is order complete and order continuous [40, p.7].

It can also be shown that a Banach lattice \( X \) is order continuous if and only if for every \( y, z \in X \), the order interval \([y, z] = \{ x : y \leq x \leq z \} \) is weakly compact [40, p.28].

Recall that a Banach space \( X \) is called weakly sequentially complete if every weak Cauchy sequence in \( X \) converges weakly to an element of \( X \).

For any Banach lattice \( X \), the following conditions are equivalent:
(i) \( X \) is weakly sequentially complete.
(ii) No subspace of \( X \) is isomorphic to \( c_0 \).
(iii) Every norm-bounded increasing sequence in \( X \) has a strong limit [40, p.34].

1.2.3. Kőthe function Spaces

Let \( (\Omega, \Sigma, \mu) \) be a complete finite measure space. A Banach space \((E, \| \cdot \|_E)\) consisting of equivalence classes, modulo equality almost everywhere, of integrable real valued functions on \( \Sigma \) is called a Kőthe function space (Banach function space) if the following conditions hold:

(i) If \( |f(\omega)| \leq |g(\omega)| \) a.e. on \( \Omega \), with \( f \) measurable and \( g \in E \), then \( f \in E \) and \( \|f\| \leq \|g\| \).

(ii) For every \( A \in \Sigma \), the characteristic function \( \chi_A \) of \( A \) belongs to \( E \) [40, p.28].

It is well known that if \((E, \| \cdot \|_E)\) is a Kőthe function space over \((\Omega, \Sigma, \mu)\) then \( L_\infty \subset E \subset L_1 \), where the inclusion maps are continuous.

Let \( L_0 \) denote the space of all \( \mu \)-equivalence classes of \( \Sigma \)-measurable real valued functions. Let \( E' \) be the Kőthe dual of \( E \) where \( E' \) is defined by

\[
E' = \{ v \in L_0 : \int_\Omega u(\omega) v(\omega) \, d\mu < \infty, \text{ for all } u \in E \}.
\]
Then the associated norm \( \| \cdot \|_{E'} \) on \( E' \) is defined by

\[
\| v \|_{E'} = \sup \{ \int_{\Omega} u(\omega) v(\omega) \, d\mu : u \in E, \| u \| \leq 1 \}.
\]

The space \( E' \) is a proper normed subspace of \( E^* \). The spaces \( E' \) and \( E^* \) coincide if and only if \( E \) is order continuous [40, p.29].

1.2.4. Integration

Let \((\Omega, \Sigma, \mu)\) be a complete finite measure space.

A function \( f : \Omega \to X \) is called simple if there exists \( x_1, x_2, \ldots, x_n \in X \) and \( E_1, E_2, \ldots, E_n \in \Sigma \) such that \( f = \sum_{i=1}^{n} x_i \chi_{E_i} \), where \( \chi_{E_i}(\omega) = 1 \) if \( \omega \in E_i \) and \( \chi_{E_i}(\omega) = 0 \) if \( \omega \notin E_i \).

A function \( f : \Omega \to X \) is called \( \mu \)-measurable (or simply measurable) if there exists a sequence of simple functions \( \{ f_n \} \) with \( \lim_{n} \| f_n - f \| = 0 \) \( \mu \)-almost everywhere (\( \mu \)-a.e.).

A \( \mu \)-measurable function is also called strongly measurable.

A function \( f : \Omega \to X \) is called weakly \( \mu \)-measurable (or scalarly \( \mu \)-measurable) if for each \( x^* \in X^* \) the numerical function \( x^* f \) is \( \mu \)-measurable.

It is well known that

**Theorem A** (Pettis's measurability theorem) [16, Theorem 2, p.42]. A function \( f : \Omega \to X \) is \( \mu \)-measurable if and only if

(i) \( f \) is \( \mu \)-essentially separably valued, that is, there exists \( E \in \Sigma \) with \( \mu(E) = 0 \) and such that \( f(\Omega \setminus E) \) is a (norm) separable subset of \( X \), and

(ii) \( f \) is weakly \( \mu \)-measurable.
A \( \mu \)-measurable function \( f : \Omega \to X \) is called Bochner integrable if there exists a sequence of simple functions \( \{ f_n \} \) such that
\[
\lim_{n \to \infty} \int_\Omega \| f_n - f \| d\mu = 0.
\]
In this case, \( \int f d\mu \) is defined for each \( E \in \Sigma \) by
\[
\int_E f d\mu = \lim_{n \to \infty} \int_E f_n d\mu \quad [16, \text{p.44-45}],
\]
which is an element of \( X \).

Bochner integrable functions are characterized by the following result:

**Theorem B** [16, Theorem 2, p.45]. A \( \mu \)-measurable function \( f : \Omega \to X \) is Bochner integrable if and only if
\[
\| f \| d\mu < \infty.
\]

Suppose \( f \) is a weakly \( \mu \)-measurable function on \( \Omega \) and \( x^* f \in L_1(\mu) \) for each \( x^* \in X^* \). Then by an application of Banach's closed graph theorem it can be shown that for each \( E \in \Sigma \) there exists \( x^{**}_E \in X^{**} \) satisfying
\[
x^{**}_E(x^*) = \int_E x^* f d\mu \quad \text{for all } x^* \in X^*.
\]
In this case, \( f \) is called Dunford integrable and the Dunford integral of \( f \) over \( E \in \Sigma \) is defined by the element \( x^{**}_E \) of \( X^{**} \) such that
\[
x^{**}_E(x^*) = \int_E x^* f d\mu \quad \text{for all } x^* \in X^* \quad \text{and we write}
\]
\[
x^{**}_E = (D) - \int_E f d\mu.
\]
If \( (D) - \int_E f d\mu \in X \) for each \( E \in \Sigma \), then \( f \) is called Pettis integrable and we write \( (P) - \int_E f d\mu \) to denote the Pettis integral of \( f \) over \( E \in \Sigma \).

By the same closed graph argument it can be shown that if \( f : \Omega \to X^* \) is a function such that \( x f \in L_1(\mu) \) for all \( x \) in \( X \), then for each set \( E \in \Sigma \) there is a vector
\( x_\epsilon^* \) in \( X^* \) such that

\[
x_\epsilon^*(x) = \int_E f \, d\mu \quad \text{for all } x \in X.
\]

The element \( x_\epsilon^* \) is called the Gelfand (or weak*) integral of \( f \) over \( E \).

It is obvious that the Dunford and Pettis integral coincide when \( X \) is reflexive.