CHAPTER 4

NATURAL FREQUENCIES OF INITIALLY LOADED STRUCTURAL MEMBERS

Prediction of free vibration characteristics like the natural frequencies and corresponding mode shapes of any structural member, are necessary for carrying out the, further investigation of the dynamic stability analysis. Evaluation of frequencies of initially loaded structural members forms an important input to study the dynamic stability behavior of these structural members when the effective stiffness is used. The effect of the initial loads (or stresses) on optimized, based on strength criterion, modern rocket and missile structural members is significant, as these operate at low margins of safety and severe service conditions. It is necessary to study the effect of these initial loads, particularly if the loads are compressive, on the free vibration characteristics of these structural members. In this chapter a simple master formula is derived to predict the free vibration frequencies of typical structural members.

Prediction of the vibration characteristics of structural members like beams, plates and shells, in particular, the evaluation of the fundamental frequency, as the energy required to excite the first mode is very less, is important to the designers in the preliminary phase of design. The exact vibration analysis of structural members with the initial loads is involved and the application of approximate solution is obtained using either the Rayliegh-Ritz (RR) method or the FE method. These methods to obtain the solutions are tedious, because of cumbersome algebraic manipulations in the RR method or larger computational times required in the case of the FE method, when a parametric study is involved. In this context, a simple and reliable master formula, if developed, will be very useful to the practicing engineers to have quick and accurate solutions to the fundamental frequency of initially loaded structural members.

Based on the earlier research works of Amba-Rao [75], Galef [76] proposed an intuitive formula to evaluate the vibration frequencies of specific case of initially loaded beams that the formula is exact, if the buckling and vibration mode shapes are similar. Bokaian [14] has shown that the Galef's [76] proposed formula is valid for certain cases of boundary conditions of beams and for the other type of boundary conditions, that the formula is either approximate or not valid. A similar conclusion about the frequency parameter corresponding to an edge radial load for circular plates with or without elastic edge rotational restraints can be seen in the works of Lurie, Singa Rao and Amba-Rao [81-83]. In the above studies, there is confusion about the definition of the frequency parameter and whether the curves between the frequency parameter and the initial load parameter are linear. Singa Rao and

Amba-Rao [83] have shown that by proper definition of the frequency parameter, these curves are more or less linear, with respect to the edge elastic restraint parameter, based on the results obtained in their study. Though there is no mention of any formula, the authors concluded that the curve is a master curve for the circular plate with elastic edge restraints, for any value of the elastic edge restraint. The similar kind of studies presented by Galef [76] and Singa Rao and Amba-Rao [83] are limited to only uniform beams and circular plates with elastic restraints. Further studies on the evaluation of the vibration frequencies of the structural members, like plates and shells with complicating effects, with compressive loads can be seen in the works of Gorman[77], Chen and Fung [126], Yang and Fu [95], Leissa [88], Rao and Neetha [11,16 and 85].

Emam and Nayfeh [78], studied the post-buckling and free vibrations of composite beams, where in the suitable non-dimensional parameters for the buckling load and the frequency parameters of composite beams are presented. The effect of the axial load on the vibration of the fundamental frequency is investigated for the pre and post buckling domains of the composite beams. However, the existence of a master formula for evaluation of free vibration frequencies of initially loaded structural members is not recognized by the researchers till now.

The main aim of the study presented in this chapter is to develop a master formula to predict the free vibration frequencies of the commonly used initially loaded structural members under compressive axial loads for beams or uniform compressive edge loads for plates and shells. Such a formula is systematically derived with mathematical rigor, based on the general matrix equilibrium equation governing the vibrations of initially loaded structural members. This formula is of the similar form proposed by Galef [76], intuitively for beams. In the development of the formula, conditions for its applicability are obtained through a condition imposed on the buckling and vibration mode shapes, contrary to the earlier mentioned intuitive formula for beams.

4.1 DEVELOPMENT OF MASTER FORMULA

Application of the classical multi-term RR or the numerical FE method to the initially loaded vibration problem of any structural member, finally gives the following matrix equilibrium equation

$$[K]\{\delta\} - \lambda_i [G]\{\delta\} - \lambda_f [M]\{\delta\} = 0 \qquad (4.1)$$

where λ_i is the axial or in-plane initial load parameter defined as

$$\lambda_i = \frac{N_i L^2}{EI} \text{ for isotropic beams,}$$

$$\lambda_i = \frac{N_i L^2}{b(D_{11} - \frac{B_{11}{}^2}{A_{11}})} \quad \text{for composite beams,}$$

$$\frac{N_i a_r^{\,2}}{D} \text{ for circular plate}$$

and $\frac{N_i A^2}{\pi^2 D}$ for square plate

and

λ_f is the initially loaded frequency parameter defined as

$$\lambda_f = \frac{\overline{m}\,\omega^2 L^4}{EI} \text{ for isotropic beams,}$$

$$\lambda_f = \frac{\overline{m}\,\omega^2 L^4}{b(D_{11} - \frac{B_{11}^{\,2}}{A_{11}})} \text{ for composite beams,}$$

$$\lambda_f = \frac{\rho h \omega^2 A^4}{D} \text{ for plates}$$

$$\lambda_f = \frac{\rho \omega^2 L^4}{A^o{}_{22}\ h} \text{ for shells}$$

and { δ} is the vibration mode shape (eigenvector).

where D is the flexural rigidity of plate, N_i is the initial load, *A* is the side of the square plate, 'a_r' is the radius of the circular plate, L is the length of the beam/shell, ω is the radian frequency with initial load, h is the thickness of the plate, A_{11}, B_{11}, D_{11} and $A^o{}_{22}$ are the extension-extension coupling coefficient, bending-extensional coupling coefficient, bending-bending coupling coefficient and extensional stiffness coefficient respectively.

The order of the matrices in Eq. (4.1) is lower, if multi-term admissible functions are used in the RR method compared to the very high matrix orders, depending on the number of elements taken, in the FE method, to achieve the desired accuracy of the frequency parameters. If a single term admissible function is used in the RR analysis, Eq. (4.1) becomes a simple scalar equation and the calculation of λ_f is straight forward. Otherwise, Eq.(4.1) is a standard matrix eigenvalue problem, containing a parameter λ_i , and can be solved to evaluate the values of λ_f with varying λ_i for any given structural member. It will be seen later that the variation λ_f with λ_i is linear if the non-dimensional parameters are properly defined such that the non-dimensional frequency and the initial load or buckling load parameters contain ω^2 and N_i *or* N_{cr} respectively. Then, the plot between the frequency parameter and the initial load parameter are obviously straight lines, but vary for different structural members. These different plots can be transformed in to a single master linear plot, applicable for all the structural members including the complicating effects, representing the variation of λ_f with respect to λ_i, obtained through the following analysis:

The degenerate case of Eq. (4.1) for the free vibration problem with out initial load, is written as,

$$[K]\{\delta_1\} - \lambda_{fo}[M]\{\delta_1\} = 0 \tag{4.2}$$

where $\{\delta_1\}$ is the mode shape of free vibration without the initial load, and λ_{fo} is the frequency parameter with out initial load (defined as $\lambda_{fo} = \frac{\overline{m}\,\omega_o^{\,2} L^4}{EI}$ for isotropic beams, $\lambda_{fo} = \frac{\overline{m}\,\omega_o^{\,2} L^4}{b(D_{11} - \frac{B_{11}^{\,2}}{A_{11}})}$ for composite beams, $\frac{\rho h \omega_o^{\,2} A^4}{D}$ for plates and $\frac{\rho \omega_o^{\,2} L^4}{A^o{}_{22}\; t}$ for shells).

where ω_o is the radian frequency without initial load.

Equation (4.2) can be rewritten as

$$[M]\{\delta_1\} = \frac{1}{\lambda_{fo}}[K]\{\delta_1\} \tag{4.3}$$

similarly, the matrix equation for the buckling problem alone can be written from Eq. (4.1), neglecting third term, as

$$[K]\{\delta_2\} - \lambda_b [G]\{\delta_2\} = 0 \tag{4.4}$$

where $\{\delta_2\}$ is the mode shape for buckling and λ_b is the buckling load parameter (defined as $\lambda_b = \frac{NL^2}{EI}$ for beams, $\lambda_b = \frac{NL^2}{b(D_{11} - \frac{B_{11}^{\,2}}{A_{11}})}$ for composite beams, $\frac{Na_r^{\,2}}{D}$ and $\frac{NA^2}{\pi^2 D}$ for the isotropic circular and square plates respectively)

or

$$[G]\{\delta_2\} = \frac{1}{\lambda_b}[K]\{\delta_2\} = 0 \tag{4.5}$$

With the condition that the buckling and vibration mode shapes is the same, i.e.

$$\{\delta_1\}=\{\delta_2\}=\{\delta\} \tag{4.6}$$

Substituting, Eqs. (4.3), (4.5) and (4.6) in Eq.(4.1) and can be written as

$$[K]\{\delta\}-\frac{\lambda_i}{\lambda_b}[K]\{\delta\}-\frac{\lambda_f}{\lambda_{f_0}}[K]\{\delta\}=0 \tag{4.7}$$

or

$$\left\{[K]\lambda_{f_0}-\lambda_{f_0}\frac{\lambda_i}{\lambda_b}[K]-\lambda_f[K]\right\}\{\delta\}=0 \tag{4.8}$$

Since $\{\delta\}\neq 0$, Eq. (4.8) can be written, as

$$\frac{\lambda_i}{\lambda_b}+\frac{\lambda_f}{\lambda_{fo}}=1 \tag{4.9}$$

From the above Eq. (4.9), it is clear that a plot between $\frac{\lambda_f}{\lambda_{fo}}$ and $\frac{\lambda_i}{\lambda_b}$ is linear and is same for any structural member. Equation (5.9) is general, and as no assumption is made on the mode of vibration, is valid for any mode for all the structural members, provided the buckling and vibration modes are the same or similar with a little difference. Further more Eq. (4.9) is satisfied exactly in the RR method if a single term admissible function is used depending on the structural members. This generally happens for the structural members with the simply supported boundary conditions. For other boundary conditions, the vibration and buckling mode shapes are similar but differ very little, and Eq. (4.9) results in a tolerably small error. However, this general equation can be effectively used, for all practical purposes, with in the accuracy acceptable by the engineering community. The evaluation of the frequency parameter of the initially loaded structural members, not only for the first mode but also for the higher modes of vibration, is discussed in detail in the following section. The aim of this study is to evaluate the frequency parameter λ_f of initially loaded structural members. The values of the parameters λ_{fo} and λ_b, for applying in the developed formula, can be obtained, for most of the structural members with complicating effects, from Hurty and Rubinstein [80] or Leissa [88] and Timoshenko and Gere [1] or Handbook of Structural Stability [2]. Otherwise, if not available in the Literature, these parameters can be obtained using the RR or FE methods.

4.2 DISCUSSION ON MASTER FORMULA

The simple formula, developed in this chapter i.e. Eq. (4.9), can be referred to as a master formula to predict the frequency parameter of any initially loaded structural member, provided the non-dimensional parameters are properly defined as discussed earlier. The only condition is to be satisfied that the mode shapes of the buckling and vibration are the same or nearly the same. This important generalization is missed by the earlier researchers on this topic by restricting their similar conclusions on beams by Amba- Rao [75] and Galef [76] and circular plates by Lurie [81] and Singa Rao and Amba-Rao [82-83] only and not extended to other structural members. Rao and Neetha [16, 45 and 85], though they have studied different problems, has not recognized the significance of the generality of this formula as the concentration is to accurately determine the values of λ_f .

The FE formulation uses a two noded element with two degrees of freedom per each node, such as lateral deflection and rotation. However, it can be observed from Gupta *et al.* [117-118] that the chosen single term assumed functions can give reasonably accurate results for both the vibration and buckling problems, where the mode shapes are assumed to be the same form in analytical studies for general set of boundary conditions and these works by Emam and Nayfeh [78], Gunda *et al.* [117] and Gupta derived in this chapter, where the mode shapes are assumed to be same for the buckling and the vibration problems.

4.3 NUMERICAL RESULTS AND DISCUSSION

The usefulness of the master formula derived in Eq. (4.9), to evaluate the fundamental frequency parameter of the structural members with initial compressive load, the results for a number of typical standard structural problems are considered in this chapter. The problems considered are slender isotropic beams on elastic foundation or with elastic end restraints [45], short tapered isotropic beams with linearly varying diameter or depth with different boundary conditions [16], slender composite beams [78,117 and 118], moderately thick simply supported and clamped circular plates [85], thin simply supported square plates subjected to uniaxial or biaxial compression [88] and a clamped composite cylindrical shell problem with end uniformly distributed load [95]. It is to be noted here, that the present study, the commonly occurring compressive loads considered are either concentrated axial end loads in the case of beams or uniformly distributed edge loads in the case of plates and shells. In these studies [88, 16, 45 and 95], the values of λ_f are evaluated independently for each problem considered, using another form of the master formula given in Eq. (4.9), as

$$\lambda_f = \lambda_{fo}\left[1 - \frac{\lambda_i}{\lambda_b}\right] \tag{4.10}$$

as the significance and importance of the master formula is not recognized in the above studies. However, the emphasis given in these studies is to find out how accurately the initially loaded fundamental frequency parameter of these structural members can be determined by using Eq. (4.10).

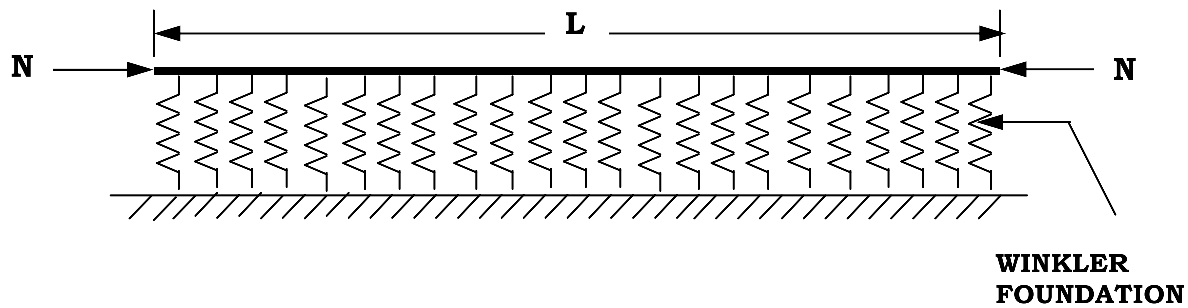

Fig. 4.1a Uniform isotropic beam on elastic foundation

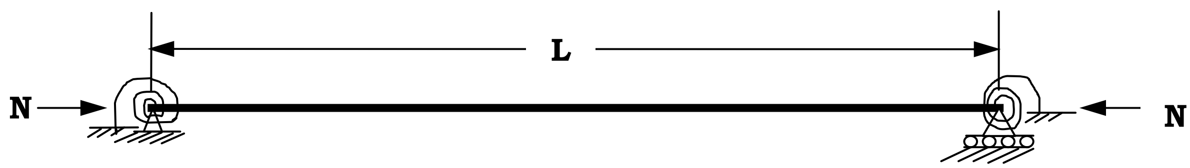

Fig. 4.1b Uniform isotropic beam with elastically restrained ends against rotation

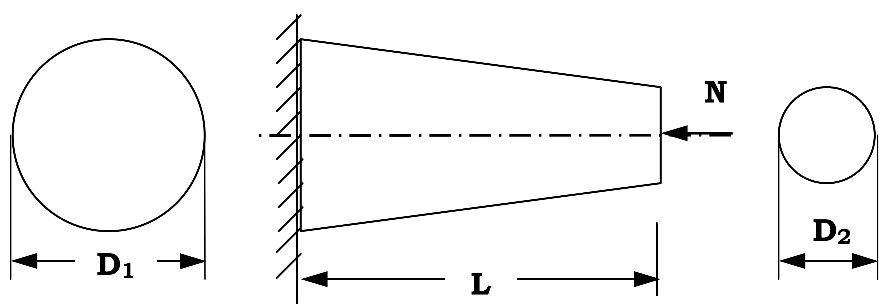

Fig.4.2a Short tapered cantilever beam of circular cross-section

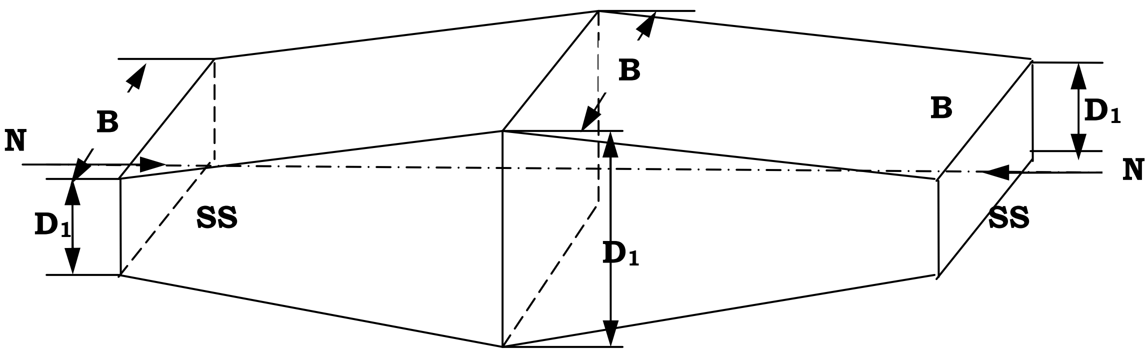

Fig. 4.2b Short symmetrically tapered simply supported (SS) beam of rectangular Cross – Section

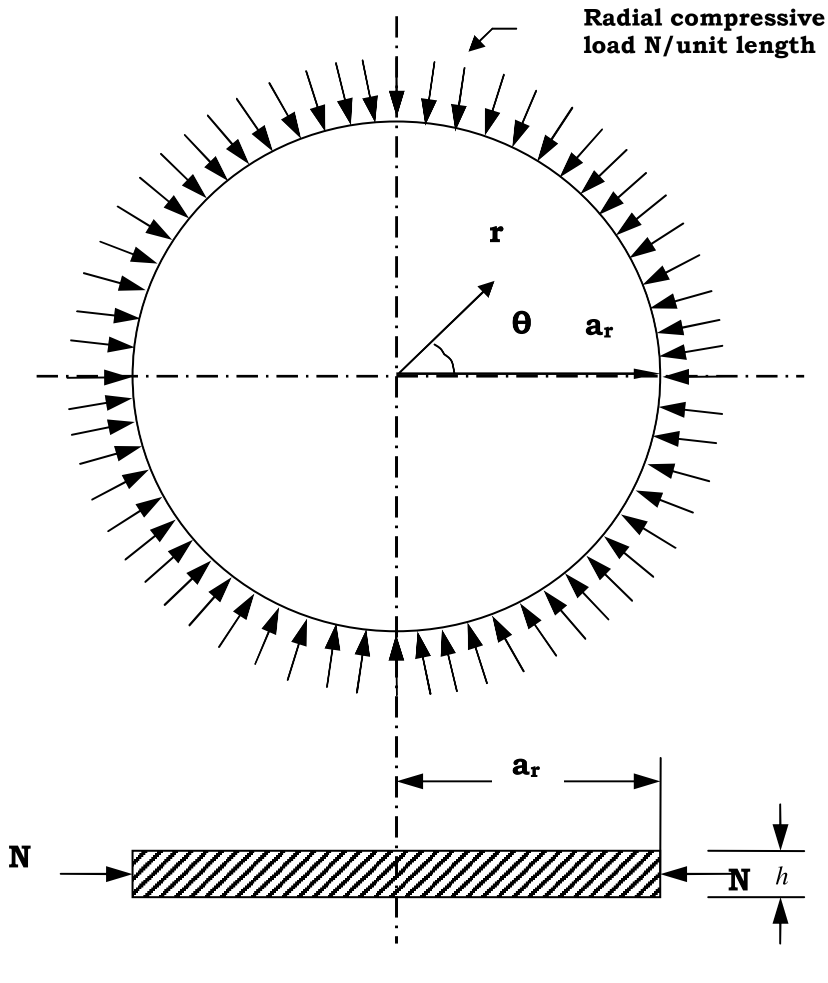

Fig. 4.3 Thick circular plate

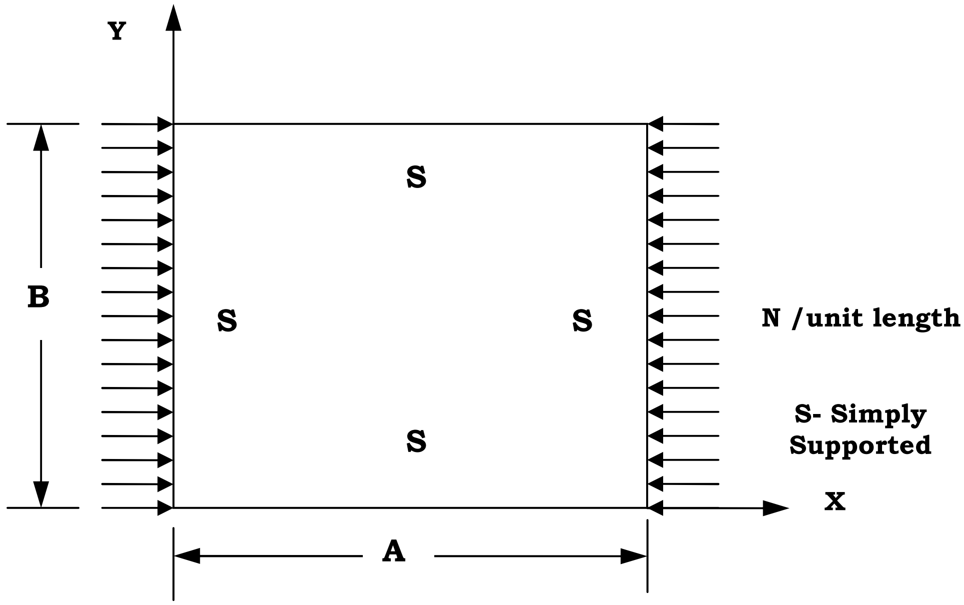

Fig. 4.4a Thin simply supported square plate under uniaxial compression

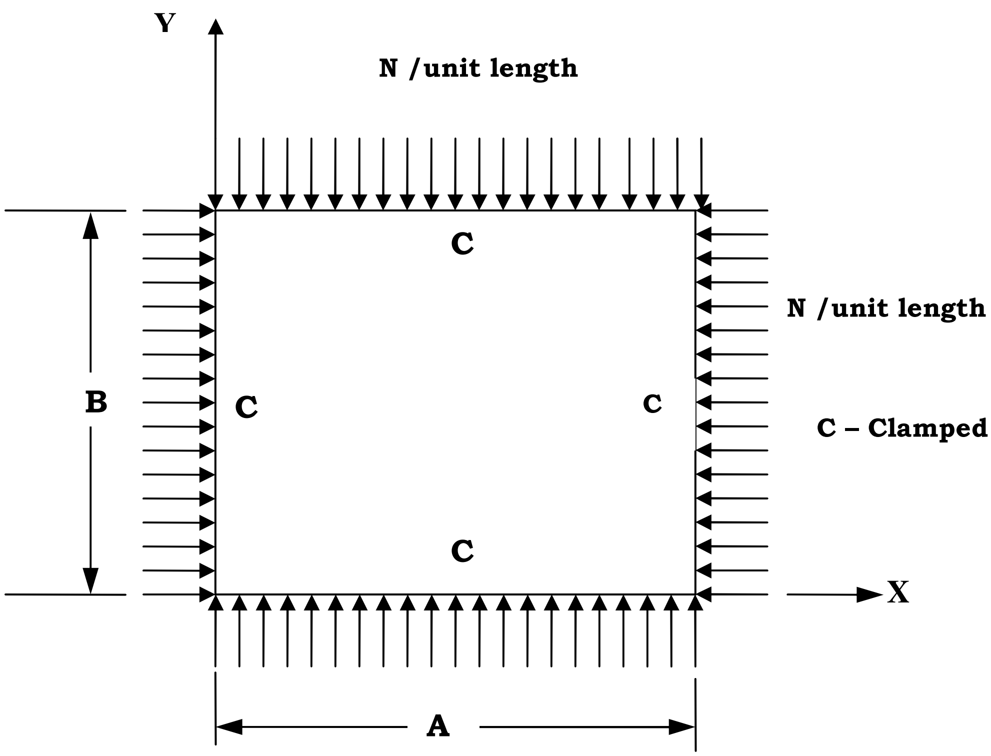

Fig. 4.4b Thin clamped square plate under biaxial compression

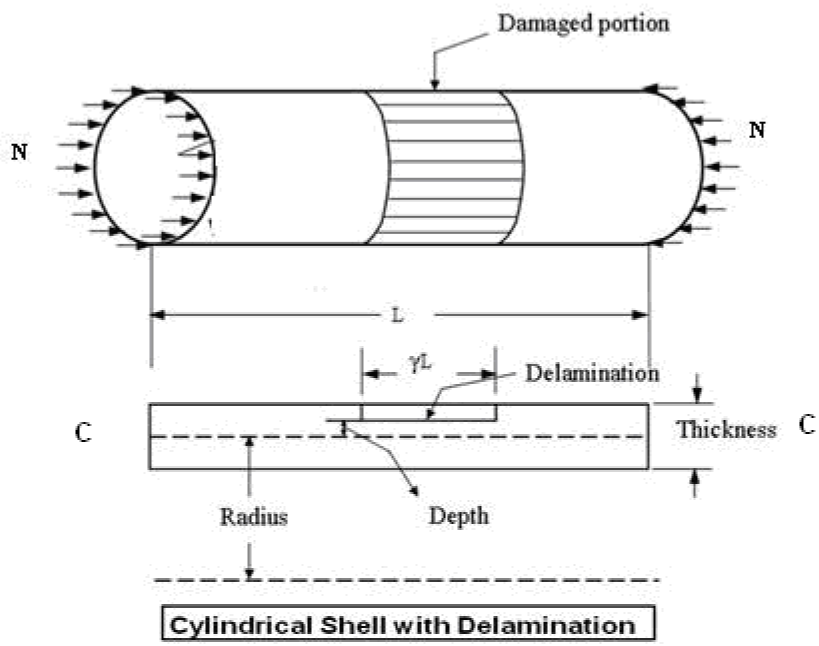

Fig. 4.5 Layered cylindrical shell with delamination under uniform end load

The values of initially loaded frequency parameter λ_f, for all the structural members considered in Figs. 4.1 to 4.5, are given in the analogue form, against the values of the initial load parameter λ_i in Figs. 4.6 to 4.11 respectively. From these figures, it can be seen, that all the plots are linear, consistent with Eq. (4.10), and are different with respect to the various parameters involved specific to the structural members considered in this chapter.

All these plots given in Figs.4.6 to 4.11 for the various structural members become a single linear plot as shown in Fig.4.12, if the proper non-dimensional parameters are used in the same way as given in Eq.

(4.9). The formula given by this equation can be treated as the master formula for obtaining the fundamental frequency parameter of the initially loaded structural members containing several specific parameters.

4.4 EXAMPLES FOR HIGHER MODES OF VIBRATION

Three particular initially loaded vibration problems for studying the higher modes of vibration of (1) simply supported beams (problem 1), (2) simply supported beam on elastic foundation or with end elastic restraints (problem 2) and (3) thin rectangular plates subjected to uniaxial or biaxial compression (problem 3) are considered in the present investigation, to show the effectiveness of the master formula. Though the complete mathematical treatment is not given here, as that is available in the standard books [1, 2, 80 and 88], only the information required appreciating the usefulness of the formula is presented.

The salient features, related to the buckling and vibration mode shapes, for the use of the formula are given here. The expressions for the relevant non-dimensional parameters involved are concisely presented in Table 4.1, for the three problems considered. For a better presentation of the results, the master formula is represented as

$$F = 1 \tag{4.11}$$

where

$$F = \frac{\lambda_f}{\lambda_{f_0}} + \frac{\lambda_i}{\lambda_b} \tag{4.12}$$

The admissible functions taken in the present study is

$$w = a \sin \frac{m\pi x}{L} \tag{4.13}$$

for beams, where the integer m represents the number of half sine waves in the x-direction and

$$w = a \sin \frac{m\pi x}{A} \sin \frac{n\pi y}{B} \tag{4.14}$$

for plates, where the integer m has the same definition as in the case of beams and n is the number of half sine waves in the y-direction. Please note that, these functions are exact for both the buckling and vibration problems of uniform simply supported beams and rectangular plates. It is to be noted here that the vibration mode shape m = 1 for the problems (2) and (3) respectively. However, m = 2 is the mode shape for the buckling problem, where the least buckling load occurs, when the foundation parameter is greater than 4.0 in the case beams on elastic foundation. Similarly, m = 2, n = 1 is the mode shape for the least buckling load for the rectangular plate with the value of the aspect ratio $\frac{A}{B}$ is 2.0. While using the master formula such peculiar behavior of the structural members has to be known to the users. Table 4.1 clearly shows that the master formula is applicable exactly when the mode shapes of the buckling and the vibration problems are exactly the same.

The formula developed in this chapter and its usefulness to predict the frequency parameters of not only the first mode of vibration but also for higher modes of vibration is conclusively demonstrated through a number of initially loaded structural members, varying from beams to shells with complicating effects. The formula developed in this study and evaluation of natural frequencies for initially loaded structural members are useful to further study the dynamic stability of structural members. The formula needs the values of load free frequency parameter and the corresponding buckling load parameter for the same mode of vibration. Care has to be taken that the frequency and buckling load parameters are properly defined. In other words, the frequency parameter should contain the ω^2 term and the initial buckling load parameters should contain N_i term in the master formula.

4. 5 CONCLUDING REMARKS

A master formula, to predict the initially loaded frequency parameters of structural members, is developed in this chapter. The efficacy of the formula, to predict the frequency parameters of not only the first mode of vibration but also for higher modes of vibration is conclusively demonstrated through a number of initially loaded structural members, varying from beams to shells with complicating effects.

Table 4.1 Results for higher modes of vibration

Problem description	Number of half sine waves	$\frac{A}{B}$	$\gamma_F = \frac{k_s L^4}{\pi^4 EI}$	λ_f	λ_{fo}	λ_b	F
1.Simple beam	Any integer $\geq 1, 1^{\wedge}$	-	-	$m^4\pi^4 - \lambda_i m^2\pi^2$	$m^4\pi^4$	$m^2\pi^2$	$=1$
2. Beam with Elastic foundation	1, 1	-	3	$4\pi^4 - \lambda_i \pi^2$	$4\pi^4$	$4\pi^2$	$=1$
	1, 2	-	5	$6\pi^4 - \lambda_i \pi^2$	$6\pi^4$	$\frac{21}{4}\pi^2$	$\neq 1$
	2, 2	-	5	$21\pi^4 - 4\lambda_i \pi^2$	$21\pi^4$	$\frac{21}{4}\pi^2$	$=1$
3.Thin rectangular plate	1, 1	1	-	$4 - \lambda_i$	4	4	$=1$
	2, 2	2	-	$64 - 4\lambda_i$	64	16	$=1$
	1, 1	2	-	$25 - \lambda_i$	25	25	$=1$
	1, 2	2	-	$25 - \lambda_i$	25	16	$\neq 1$

^ First and second integers represent the number of half sine waves of vibration and buckling modes in the x- and y-directions respectively.

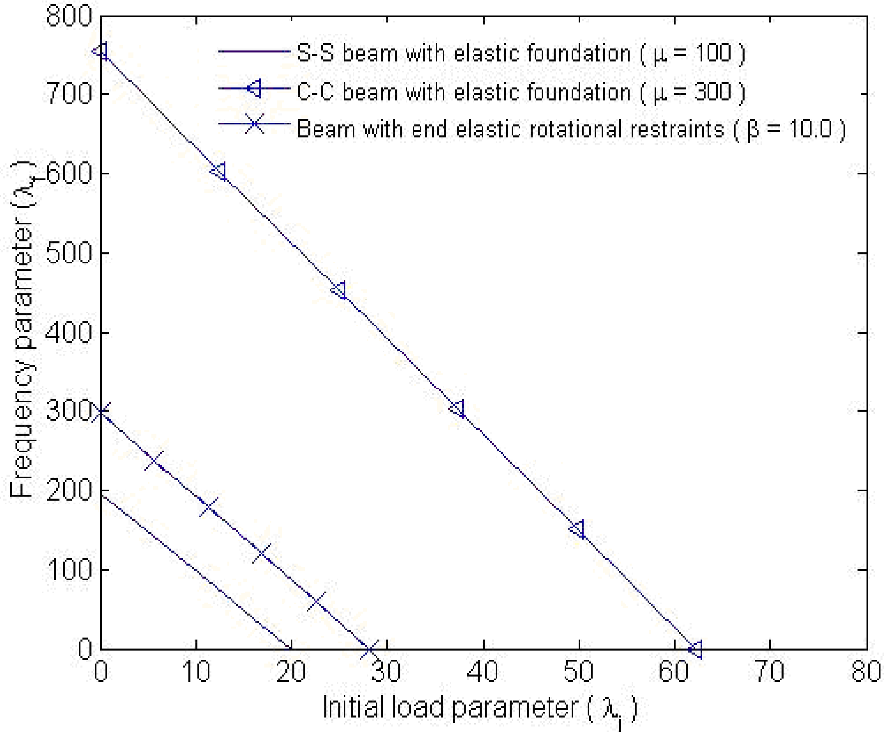

Fig. 4.6 Slender uniform beams with elastic foundation and elastically restrained against rotation

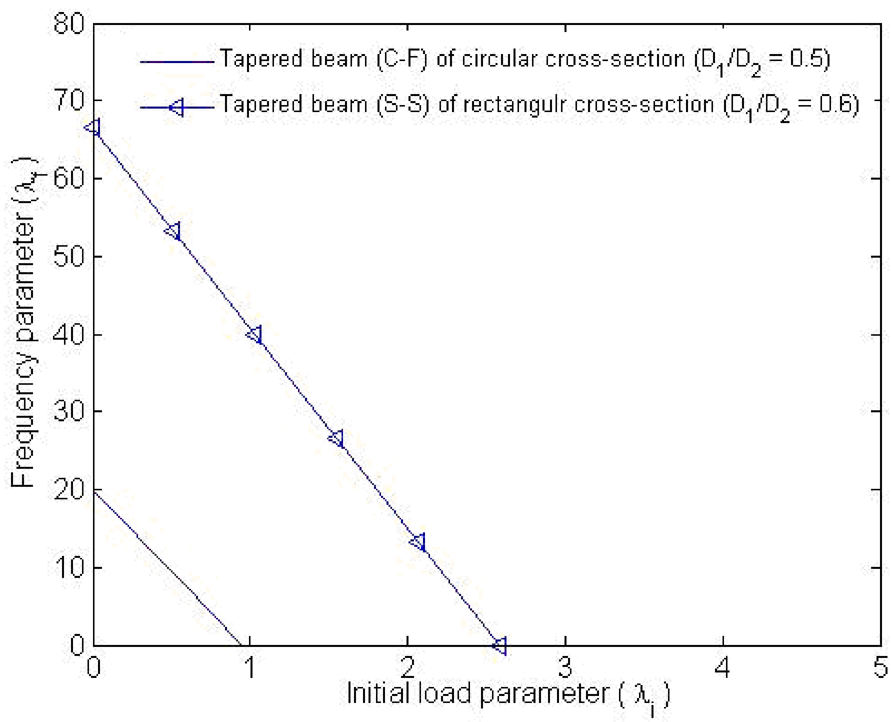

Fig. 4.7 Short tapered beams of circular (clamped-fixed) and rectangular cross-section (simply supported)

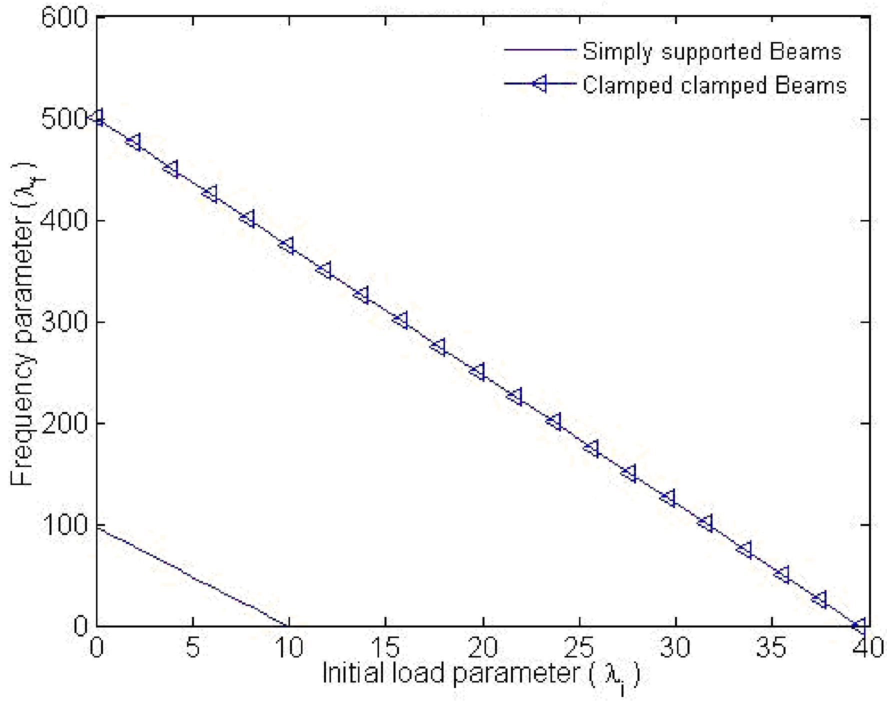

Fig. 4.8 Composite beams with simply supported (S-S) and clamped-clamped (C-C) end conditions

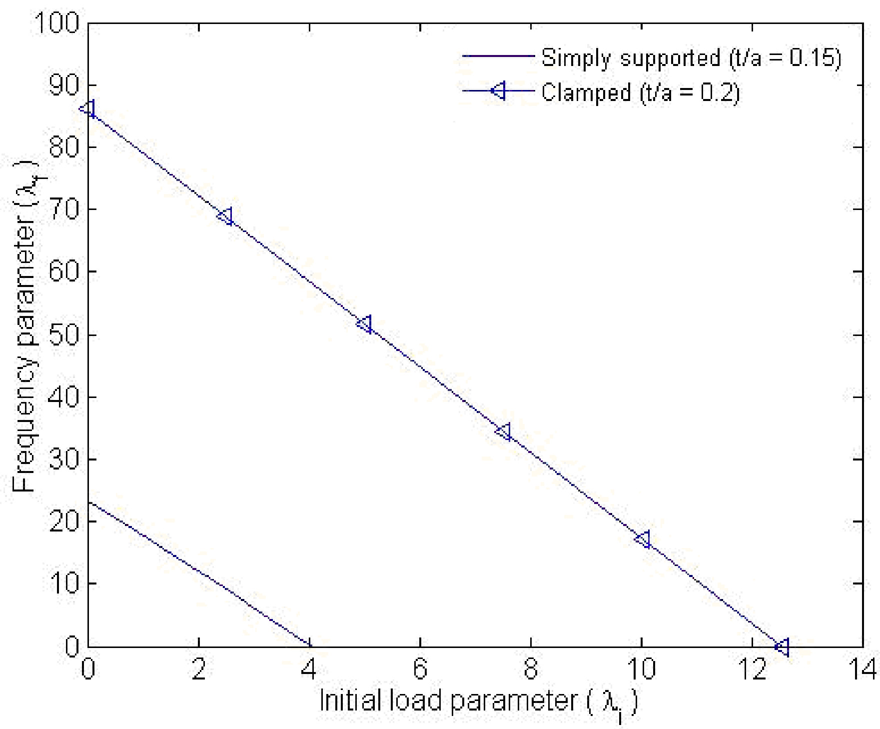

Fig.4.9. Moderately thick circular plate with radial compressive load

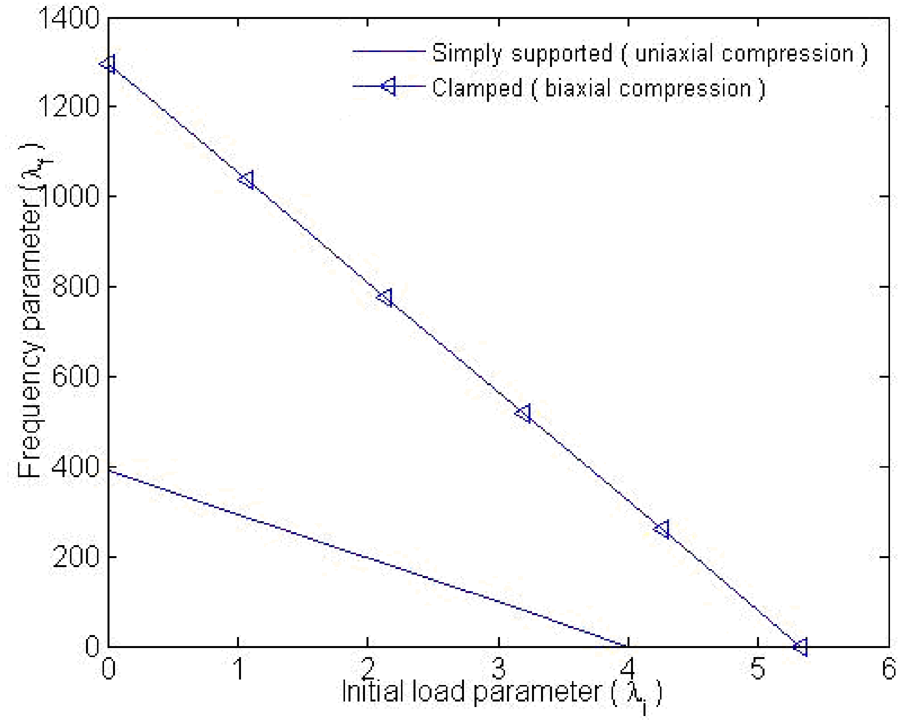

Fig. 4.10 Thin square plate with simply supported under uniaxial compression and clamped square plate under biaxial Compression

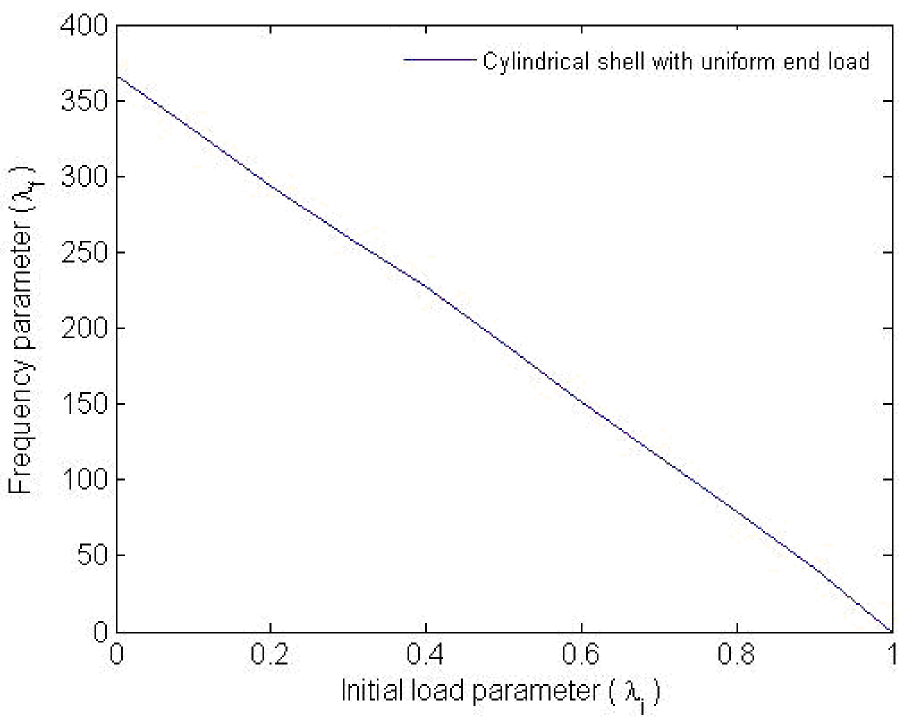

Fig. 4.11 Layered cylindrical shell with delamination under uniform end load

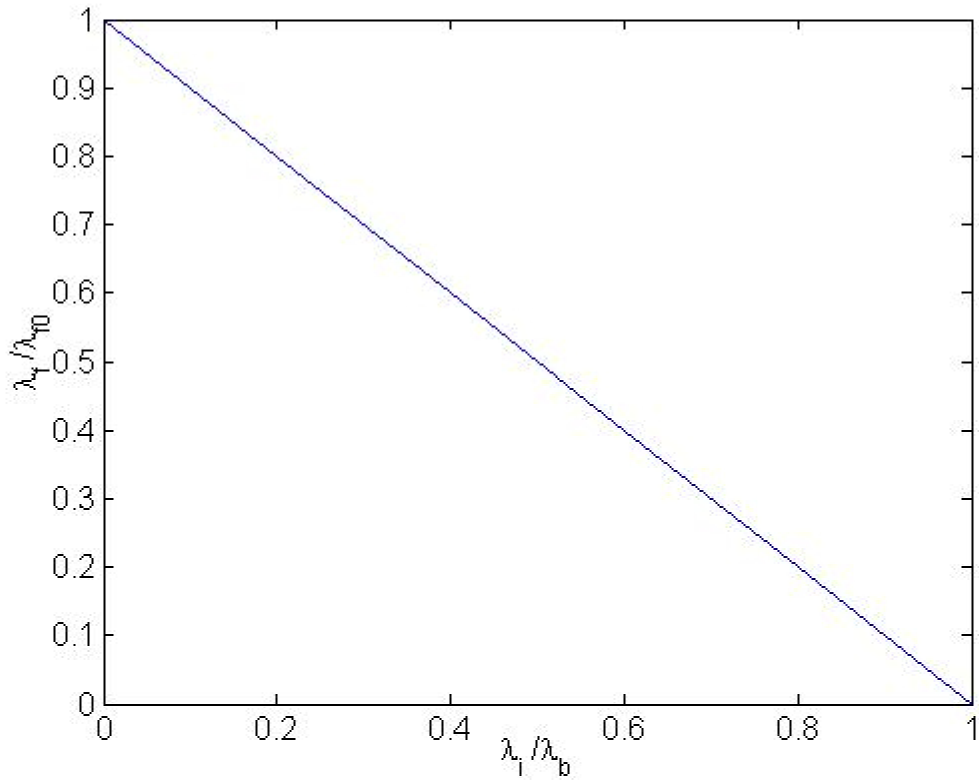

Fig. 4.12 Universal curve for initially loaded structural member