Chapter 4

On Finite Sum of Polynomial Expressions

4.1 Introduction.

For a positive integer p, to find the value of $\sum_{r=1}^{n} r^p$, there are some methods, for example Principle of Undetermined coefficient [30]:

$$\sum_{r=1}^{n} r^p = \frac{n^{p+1}}{p+1} + \frac{n^p}{2} + \frac{B_1 p n^{p-1}}{2!} - \frac{B_3 p(p-1)(p-2)n^{p-3}}{4!} + B_5 \frac{p(p-1)(p-2)(p-3)(p-4)}{6!} n^{p-5} - \cdots \tag{4.1.1}$$

where $B_1 = \frac{1}{6}, B_2 = \frac{1}{30}, B_5 = \frac{1}{42}, B_9 = \frac{1}{30}, B_9 = \frac{5}{66}, \cdots$ are Bernoulli's Numbers.
We are interested for $p = 1, 2, 3, \cdots$ upto some finite values, say $p = 7$ only. Without using any formula, in section 2, we have obtained these results in a simple manner and in very less time.
R.H.S. of equation (4.1.1) is 0 if $n = 0$ and it is 1 when $n = 1$. Obviously n is a factor of R.H.S. equation (4.1.1), which is $(p+1)^{th}$ degree polynomial in n.
Notations: Let $\sum_{r=1}^{n} g(r) = \sum g(n)$
where $g(r)$ be a polynomial in r.

$$\sum g(n) = \begin{cases} 0 & \text{if } n = 0, \\ g(1) & \text{if } n = 1, \end{cases}$$

$$\sum_{r=1}^{n} r^p = \sum n^p$$

Take $\frac{d}{dn}\sum g(n) = \sum \frac{d}{dn} g(n)$, $\quad \int \sum g(n) = \sum \int g(n)dn$

Let $\Delta g(n) = g(n) - g(n-1)$.

Then $f^{p+1}(n) = 0$ if $f(n)$ is a polynomial in n upto degree p, knowing $p+1$ different values of $f(n)$ for the values of n, we can determine $f(n)$ as a polynomial in n upto degree p by Newton forward difference interpolation formula.

By principle of mathematical induction one can easily verify the formulae obtained by equation (4.1.1)

4.2 Integral Technique.

$$\begin{aligned} \text{If } g(n) &= \text{Polynomial in } n \text{ of degree } p \\ &= a_0 + a_1 n + a_2 n^2 + \cdots + a_p n^p, \end{aligned}$$

where a_i's are known constants (real), then

$\sum g(n) =$ Polynomial of degree $p+1$ and

$\int g(n)dn = a_0 n + \frac{1}{2}a_1 n^2 + \frac{1}{3}a_3 n^3 + \cdots + \frac{1}{p+1}a_p n^{p+1} + \text{constant K}$

$$\begin{aligned} \Rightarrow \sum \int g(n)dn &= \sum (a_0 n + \cdots + \frac{1}{p+1}a_p n^{p+1}) + \sum K \\ &= (\text{Polynomial in } n \text{ of degree } p+2) + K\sum 1 \end{aligned}$$

$$\begin{aligned} \Rightarrow \sum \int g(n)dn &= \int (\text{some polynomial of degree } p+1)dn + Kn \\ &= \int f(n)dn + Kn \end{aligned}$$

where $f(n)$ is some polynomial of degree $p+1$.
"If $\sum g(n) = f(n)$" where $g(n)$ is a p -degree polynomial in n and $f(n)$ is a $p+1$ degree polynomial in n (with $f(0) = 0$), then adding K on R.H.S. and integrating with respect to n we get,

$$\sum \int g(n)dn = \int f(n)dn + Kn$$

where K is a real constant. "
In L.H.S., under summation there may be a constant of integration and R.H.S. must be 0 for $n = 0$ (that is, in particular, constant term on R.H.S. is 0).
Taking particular value of a constant of integration in L.H.S. under summation and putting $n = 1$, we can determine the value of K and hence we obtain a formula for $\sum \int g(n)dn$. In particular taking $g(n) = n^p$ where $p \geq 0$, we obtain the formulae for $\sum n^{p+1}, \quad \sum n^{p+2}, \cdots$.

4.2.1 Determination of $\sum n^p$ for $p = 1, 2, 3, \cdots$

$\sum 1 = 1 + 1 + \cdots$ upto n times. Therefore

$$\sum 1 = n \tag{4.2.1}$$

Adding constant K on R.H.S. of equation (4.2.1), integrating with respect to n and taking constant of integration 0 i.e. using integral technique,

$$\sum n = \frac{n^2}{2} + Kn$$

For $n = 1$, we have $1 = \frac{1}{2} + K$ and so $K = \frac{1}{2}$.

$$\text{Hence} \quad \sum n = \frac{n^2}{2} + \frac{n}{2} \quad \text{i.e.} \sum n = \frac{n(n+1)}{2} \tag{4.2.2}$$

Again using integral technique,

$$\begin{aligned}\sum \frac{n^2}{2} &= \frac{n^3}{6} + \frac{n^2}{4} + K_1 n \\ \text{i.e.} \sum n^2 &= \frac{n^3}{3} + \frac{n^2}{2} + 2K_1 n\end{aligned}$$

For $n = 1$, $\quad 1 = \frac{1}{3} + \frac{1}{2} + 2K_1$ and so $\quad 2K_1 = \frac{1}{6}$

$$\text{Hence} \sum n^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} \quad \text{i.e.} \sum n^2 = \frac{n(n+1)(2n+1)}{6} \tag{4.2.3}$$

Using integral technique

$$\begin{aligned}\sum \frac{n^3}{3} &= \frac{n^4}{12} + \frac{n^3}{6} + \frac{n^2}{12} + K_2 n \\ \text{i.e.} \sum n^3 &= \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4} + 3K_2 n\end{aligned}$$

For $n = 1$, $\quad 1 = \frac{1}{4} + \frac{1}{2} + \frac{1}{2} + 2K_2$ and so $\quad 3K_2 = 0$

$$\text{Hence} \sum n^3 = \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4} \quad \text{i.e.} \sum n^3 = \left[\frac{n(n+1)}{2}\right]^2 \tag{4.2.4}$$

Again using integral technique

$$\begin{aligned}\sum \frac{n^4}{4} &= \frac{n^5}{20} + \frac{n^4}{8} + \frac{n^3}{12} + K_3 n \\ \text{i.e.} \sum n^4 &= \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} + 4K_3 n\end{aligned}$$

For $n = 1$, $\quad 1 = \frac{1}{5} + \frac{1}{2} + \frac{1}{3} + 4K_3$ and so $\quad 4K_3 = -\frac{1}{30}$

Hence $\quad \sum n^4 = \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}$

$$\text{i.e.} \sum n^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} \tag{4.2.5}$$

Using integral technique

$$\sum \frac{n^5}{5} = \frac{n^6}{30} + \frac{n^5}{10} + \frac{n^4}{12} - \frac{n^2}{60} + K_4 n$$
$$\text{i.e.} \sum n^5 = \frac{n^6}{6} + \frac{n^5}{2} + \frac{5n^4}{12} - \frac{n^2}{12} + 5K_4 n$$

For $n = 1$, $\quad 1 = \frac{1}{6} + \frac{1}{2} + \frac{5}{12} - \frac{1}{12} + 5K_4$ and so $\quad 5K_4 = 0$

$$\text{Hence} \quad \sum n^5 = \frac{n^6}{6} + \frac{n^5}{2} + \frac{5n^4}{12} - \frac{n^2}{12} \tag{4.2.6}$$

Using integral technique to equation (4.2.6)

$$\sum \frac{n^6}{6} = \frac{n^7}{42} + \frac{n^6}{12} + \frac{n^5}{12} - \frac{n^3}{36} + K_5 n$$
$$\text{i.e.} \sum n^6 = \frac{n^7}{7} + \frac{n^6}{2} + \frac{n^5}{2} - \frac{n^3}{3} + 6K_5 n.$$

For $n = 1$, $\quad 1 = \frac{1}{7} + \frac{1}{2} + \frac{1}{2} - \frac{1}{6} + 6K_5$ and so $\quad 6K_5 = \frac{1}{42}$

$$\text{Hence} \quad \sum n^6 = \frac{n^7}{7} + \frac{n^6}{2} + \frac{n^5}{2} - \frac{n^3}{6} + \frac{n}{42} \tag{4.2.7}$$

Again using integral technique to equation (4.2.7)

$$\sum \frac{n^7}{7} = \frac{n^8}{56} + \frac{n^7}{14} + \frac{n^6}{12} - \frac{n^4}{24} + \frac{n^2}{84} + K_6 n$$
$$\text{i.e.} \sum n^7 = \frac{n^8}{8} + \frac{n^7}{2} + \frac{7n^6}{12} - \frac{7n^4}{14} + \frac{n^2}{12} + 7K_6 n.$$

For $n = 1$, $\quad 1 = \frac{1}{8} + \frac{1}{2} + \frac{7}{12} - \frac{7}{24} + \frac{1}{12} + 7K_6$ and so $\quad 7K_6 = 0$

$$\text{Hence} \quad \sum n^7 = \frac{n^8}{8} + \frac{n^7}{2} + \frac{7n^6}{2} - \frac{7n^4}{24} + \frac{n^2}{12}. \tag{4.2.8}$$

4.2.2 Determination of $\sum g(n)$ when $g(n)$ is a polynomial.

Applying the integral technique successively on $\sum 1 = n$, we get

$$\sum (n+a) = \frac{n^2}{2} + Kn \tag{4.2.9}$$

$$\sum \left(\frac{n^2}{2} + an + b\right) = \frac{n^3}{6} + \frac{Kn^2}{2} + K_1 n \tag{4.2.10}$$

$$\sum \left(\frac{n^3}{6} + \frac{an^2}{2} + bn + c\right) = \frac{n^4}{24} + \frac{Kn^3}{6} + \frac{K_1 n^2}{2} + K_2 n \tag{4.2.11}$$

and so on upto the degree of a polynomial in the L.H.S under summation as the degree of $g(n)$. Here $a, b, c, \cdots$ and $K, K_1, K_2, \cdots$ are constants.

Comparing L.H.S in the last step with $\sum g(n)$ we fix the values of $a, b, c, \cdots$ and using $n = 1$ in equations (4.2.9), (4.2.10), (4.2.11),$\cdots$ we find the values of $K, K_1, K_2, \cdots$ and hence we get the formula for $\sum g(n)$.

Example 4.2.1. Put $a = -\frac{1}{2}$ in equation (4.2.9).

Then $\sum (2n-1) = n^2 + 2Kn$

Taking $n = 1$, we get $1 = 1 + 2K$ and so $2K = 0$. Therefore

$$\sum (2n-1) = n^2 \tag{4.2.12}$$

Example 4.2.2. To determine $\sum n(n+1)$, take $a = \frac{1}{2}, b = 0$ in equations (4.2.9) and (4.2.10).

Then for $n = 1$, $\quad \frac{3}{2} = \frac{1}{2} + K$

i.e. $K = 1$ and $1 = \frac{1}{6} + \frac{1}{2} + K_1$ i.e. $K_1 = \frac{1}{3}$

Hence $\sum \left(\frac{n^2}{2} + \frac{n}{2}\right) = \frac{n^3}{6} + \frac{n^2}{2} + \frac{n}{3}$

i.e. $\sum n(n+1) = \frac{n(n+1)(n+2)}{3}$

Example 4.2.3. To determine $1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2$, use integral technique for equation (4.2.12),

we get, $\sum \frac{(2n-1)^2}{4} = \frac{n^3}{3} + Kn$

Taking $n = 1$, $\frac{1}{4} = \frac{1}{3} + K$ and so $K = -\frac{1}{12}$.

Hence $\sum (2n-1)^2 = 4\left[\frac{n^3}{3} - \frac{n}{12}\right] = \frac{n}{3}(4n^2 - 1)$

Example 4.2.4. To determine $\sum n(n+1)(n+2)$ i.e. $\sum (n^3 + 3n^2 + 2n)$

Take $a = 1, b = \frac{1}{3}$ and $c = 0$ in equations (4.2.9), (4.2.10), (4.2.11).

Taking $n = 1$, we get $K = \frac{3}{2}, K_1 = \frac{11}{12}$ and $K_3 = \frac{1}{4}$.

Hence equation (4.2.11) gives $\sum \left(\frac{n^3}{6} + \frac{n^2}{2} + \frac{n}{3}\right) = \frac{n^4}{24} + \frac{n^3}{4} + \frac{11n^2}{24} + \frac{n}{4}$

i.e. $\sum (n^3 + 3n^2 + 2n) = \frac{n^4}{4} + \frac{3n^3}{2} + \frac{11n^2}{4} + \frac{3n}{2}$.

Example 4.2.5. Show that $1 \cdot 2^2 + 2 \cdot 3^2 + 3 \cdot 4^2 + \cdots + n \cdot (n+1)^2$
$= \frac{n^4}{4} + \frac{7n^3}{6} + \frac{7n^2}{4} + \frac{5n}{6}$

and $\sum n(n+2)(n+4) = \frac{n^4}{4} + \frac{5n^3}{2} + \frac{29n^2}{4} + 10n$.

4.3 Differentiation Technique.

It is the reverse process of integral technique.

"If we know the formula $\sum g(n) = f(n)$ where $g(n), f(n)$ be polynomials in n of degree $p+1, p+2$ respectively, then differentiating both sides with respect to n and removing constant term on the right hand side, we get $\sum g'(n) = f'(n) - f'(0)$".

Here polynomial is L.H.S under summation is p th degree etc.

$$\text{If} \quad g(n) = n^{p+1} \quad \text{i.e.} \quad \sum n^{p+1} = f(n) \tag{4.3.1}$$

$$\text{Then} \quad \sum n^p = \frac{f'(n) - f'(0)}{p+1} \tag{4.3.2}$$

We find formulae for $\sum 1, \sum n, \sum n^2, \cdots, \sum n^p$, if we know formula for $\sum n^{p+1}$ where integer $p \geq 0$.

Comparing equation (4.2.8) with equation (4.3.1) and using equation (4.3.2) i.e. using differentiation technique to equation equation (4.2.8), we get,

$$\begin{aligned}\sum n^6 &= \frac{n^7 + \frac{7}{2}n^6 + \frac{7}{2}n^5 - \frac{7}{6}n^3 + \frac{n}{6} - 0}{7}\\ &= \frac{n^7}{7} + \frac{n^6}{2} + \frac{n^5}{2} - \frac{n^3}{6} + \frac{n}{42}, \quad \text{which is equation (4.2.7)}\end{aligned}$$

Using differentiating technique,

$$\begin{aligned}\sum n^5 &= \frac{n^6 + 3n^5 + \frac{5}{2}n^4 - \frac{1}{2}n^2 + \frac{1}{42} - \frac{1}{42}}{6}\\ &= \frac{n^6}{6} + \frac{n^5}{2} + \frac{5}{12}n^4 - \frac{1}{12}n^2, \quad \text{which is equation (4.2.6)}\end{aligned}$$

Again using differentiation technique successively,

$$\begin{aligned}\sum n^4 &= \frac{n^5 + \frac{5}{2}n^4 + \frac{5}{3}n^3 - \frac{n}{6} - 0}{5}\\ &= \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}, \quad \text{which is equation (4.2.5)}\end{aligned}$$

$$\begin{aligned}\sum n^3 &= \frac{n^4 + 2n^3 + n^2 - \frac{1}{30} - (-\frac{1}{30})}{4}\\ &= \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4}, \quad \text{which is equation (4.2.4)}\end{aligned}$$

$$\begin{aligned}\sum n^2 &= \frac{n^3 + \frac{3}{2}n^2 + \frac{n}{2} - 0}{3}\\ &= \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}, \quad \text{which is equation (4.2.3)}\end{aligned}$$

$$\begin{aligned}\sum n &= \frac{n^2 + n + \frac{1}{6} - \frac{1}{6}}{2}\\ &= \frac{n^2}{2} + \frac{n}{2}, \quad \text{which is equation (4.2.2)}\end{aligned}$$

$$\begin{aligned}\sum 1 &= \frac{n+\frac{1}{2}-\frac{1}{2}}{1}\\ &= n, \quad \text{which is equation (4.2.1)}\end{aligned}$$

Example 4.3.1. If it is given that

$$\sum (2n-1)^2 = \frac{4n^3}{3} - \frac{n}{3} \tag{4.3.3}$$

then we can determine $\sum (2n-1)$.
Applying differentiating technique to equation (4.3.3), we get

$$\sum 4(2n-1) = 4n^2 - \frac{1}{3} - \Big(-\frac{1}{3}\Big).$$

Hence $\sum (2n-1) = n^2$, which is equation (4.2.12).

4.4 Forward Difference Technique.

Newton's forward difference interpolation formula [55]:
$f(a+nh) = f(a) + n\Delta^1 f(a) + \frac{n(n-1)}{2!}\Delta^2 f(a) + \frac{n(n-1)(n-2)}{3!}\Delta^3 f(a) + \cdots$
Taking $a=0$ and $h=1$, we obtain

$$f(n) = f(0) + n\Delta f(0) + \frac{n(n-1)}{2!}\Delta^2 f(0) + \cdots \tag{4.4.1}$$

We can use equation (4.4.1) to obtain the value of $\sum_{x=1}^{n} x^p$ where p is a nonnegative integer. Value of $\sum_{x=1}^{n} x^p$ is expressed as polynomial expression in n of degree $p+1$.
For this we need $p+2$ values of $f(x)$ at $x = 0, 1, 2, \cdots, p+1$.

Table 4.1: Forward Difference table for $f(n) = 1 + 2 + 3 + \cdots + n$

x	f(x)	Δ f(x)	Δ^2 f(x)
0	0		
		1	
1	1		1
		2	
2	3		

4.4.1 To find $1 + 2 + 3 + \cdots n = f(n)$, Quadratic in n:

By equation (4.4.1),

$$f(n) = f(0) + n\Delta f(0) + \frac{n(n-1)}{2}\Delta^2 f(0)$$
$$= 0 + n + \frac{n(n-1)}{2}$$
$$= \frac{n(n+1)}{2}$$

4.4.2 To find $1^2 + 2^2 + 3^2 + \cdots + n^2 = f(n)$, third degree polynomial in n.

Table 4.2: Forward Difference table for $f(n) = 1^2 + 2^2 + 3^2 + \cdots + n^2$

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	0			
		1		
1	$1^2 = 1$		3	
		4		2
2	$1^2 + 2^2 = 5$		5	
		9		
3	$1^2 + 2^2 + 3^2 = 14$			

By equation (4.4.1),

$$f(n) = f(0) + n\Delta f(0) + \frac{n(n-1)}{2}\Delta^2 f(0) + \frac{n(n-1)(n-2)}{6}\Delta^3 f(0)$$

$$= 0 + n + \frac{3n(n-1)}{2} + \frac{2n(n-1)(n-2)}{6}$$
$$= \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} = \frac{n(n+1)(2n+1)}{6}$$

4.4.3 To find $1^3 + 2^3 + 3^3 + \cdots + n^3 = f(n)$, fourth degree polynomial in n.

Table 4.3: Forward Difference table for $f(n) = 1^3 + 2^3 + 3^3 + \cdots + n^3$

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
0	0				
		1			
1	$1^3 = 1$		7		
		8		12	
2	$1^3 + 2^3 = 9$		19		6
		27		18	
3	$1^3 + 2^3 + 3^3 = 36$		37		
		64			
4	$1^3 + 2^3 + 3^3 + 4^4 = 100$				

By equation (4.4.1), we have

$$f(n) = f(0) + n\Delta f(0) + \frac{n(n-1)}{2}\Delta^2 f(0) + \frac{n(n-1)(n-2)}{6}\Delta^3 f(0) + \frac{n(n-1)(n-1)(n-3)}{24}\Delta^4 f(0)$$
$$= 0 + n + \frac{7n(n-1)}{2} + \frac{12n(n-1)(n-2)}{6} + \frac{6n(n-1)(n-2)(n-3)}{24}$$
$$= \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4} = \left[\frac{n(n+1)}{2}\right]^2$$

4.4.4 To find $1^4 + 2^4 + 3^4 + \cdots + n^4 = f(n)$, fifth degree polynomial in n.

By equation (4.4.1),

$$f(n) = f(0) + n\Delta f(0) + \frac{n(n-1)}{2}\Delta^2 f(0) + \cdots + \frac{n(n-1)(n-1)(n-3)(n-4)}{120}\Delta^5 f(0)$$

Table 4.4: Forward Difference table for $f(n) = 1^4 + 2^4 + 3^4 + \cdots + n^4$

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$
0	0					
		1				
1	1^4		15			
		16		50		
2	$1^4 + 2^4$		65		60	
		81		110		24
3	$1^4 + 2^4 + 3^4$		175		84	
		256		194		
4	$1^4 + 2^4 + 3^4 + 4^4$		369			
		625				
5	$1^4 + 2^4 + 3^4 + 4^4 + 5^4$					

$$= 0 + n + \frac{15n(n-1)}{2} + \frac{15n(n-1)}{2} + \frac{15n(n-1)(n-2)}{6}$$
$$+ \frac{60n(n-1)(n-2)(n-3)}{24} + \frac{24n(n-1)(n-2)(n-3)(n-4)}{120}$$
$$= \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}$$

4.4.5 To find $1^5 + 2^5 + 3^5 + \cdots + n^5 = f(n)$, sixth degree polynomial in n.

By equation (4.4.1) and table 4.5,

$$f(n) = f(0) + n\Delta f(0) + \frac{n(n-1)}{2}\Delta^2 f(0) + \cdots + \frac{n(n-1)(n-1)(n-3)(n-4)(n-5)}{720}\Delta^6 f(0)$$
$$= 0 + n + \frac{31n(n-1)}{2} + \frac{180n(n-1)(n-2)}{6} + \frac{390n(n-1)(n-2)(n-3)}{24}$$
$$+ \frac{360n(n-1)(n-2)(n-3)(n-4)}{120} + \frac{120n(n-1)(n-2)(n-3)(n-4)(n-5)}{720}$$
$$= \frac{n^6}{6} + \frac{n^5}{2} + \frac{5}{12}n^4 - \frac{1}{12}n^2$$

Table 4.5: Forward Difference table for $f(n) = 1^5 + 2^5 + 3^5 + \cdots + n^5$

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$	$\Delta^6 f(x)$
0	0=0						
		1					
1	$1^5 = 1$		31				
		32		180			
2	$1^4 + 2^5 = 33$		211		390		
		243		570		360	
3	$1^5 + 2^5 + 3^5 = 276$		781		750		120
		1024		1320		480	
4	$1^5+2^5+3^5+4^5 = 1300$		2101		1230		
		3125		2550			
5	$1^5+2^5+3^5+4^5+5^5 = 1425$		4651				
		7776					
6	$1^5+2^5+3^5+4^5+5^5+6^5 = 12201$						

Example 4.4.1. Operating both sides of formula, say $\sum g(n) = f(n)$ under difference operator, we obtain a lower degree formula. For example suppose we have,

$$\sum n^2 = \frac{n(n+1)(2n+1)}{6}$$

Then $\Delta \sum n^2 = \Delta\Big[\frac{1}{6}(2n^2 + 3n^2 + n)\Big]$

i.e. $\sum[n^2 - (n-1)^2] = \frac{1}{6}\big[(2n^2 + 3n^2 + n) - (2[n-1]^3 + 3[n-1]^2 + [n-1])\big]$

$\Rightarrow \sum(2n-1) = n^2,$ which is equation (4.2.12).