Chapter 3

Triplets and Sum of Its Two Coordinates

3.1 Introduction.

Number theory has been a subject of study by mathematicians from the most ancient of times (3000 B. C.) [31]. The Greeks had a deep interest in Number Theory. Euclid's great text, The Elements, contains a fair amount of Number Theory; which includes the infinitude of the primes, determination of all primitive Pythagorean triples, irrationality of $\sqrt{2}$ etc.

A remarkable aspect of Number Theory is that there is something in it for every one from puzzles as entertainment for laymen to many open problems for scholars and mathematicians. Following are some open problems:

Goldbach Conjecture i.e. every even integer greater that 2 can be expressed as sum of two primes, Twin prime conjecture i.e. there exist infinitely many positive integers n so that both n and $n + 2$ are prime, there is no odd perfect number [2], [40].

Perfect square numbers are $1^2 = 1, 2^2 = 4, 3^2 = 9, 4^2 = 16, 5^2 = 25...$ and for any $n \in \mathbb{N} = \{1, 2, 3, 4, ...\}$, any positive integer k, with $n^2 < k < (n+1)^2$, is not a perfect

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square.

$k-$ tuple of positive integers $(k \geq 3)$, for any $n \in \mathbb{N}, (2n^2, 2n^2, \cdots, 2n^2)$ is such that sum of any two coordinates is $4n^2$, a perfect square. Such $k-$tuples are infinitely many. For any $m, n \in \mathbb{N}$ with $m^2 > 2n^2, m \neq 2n, k \geq 3$, the $k-$tuple $(m^2-2n^2, 2n^2, 2n^2, \cdots, 2n^2)$ is such that sum of its any two coordinates is a perfect square. Such $k-$ tuples are infinitely many and first coordinate is different from other coordinates. Above are trivial examples of $k-$ tuples in which sum of any two coordinates is a perfect square.

$k-$ tuple of positive integers $(k \geq 3)$, for any $m, n \in \mathbb{N}, (n \geq 2)$,

$(2^{n-1}m^n, 2^{n-1}m^n, \cdots, 2^{n-1}m^n)$ is such that the sum of any two coordinates is $(2m)^n$, nth power of the positive integer $2m$. Taking $m = 1, 2, 3, \cdots$, we get such infinitely many $k-$tuples.For any $p, q \in \mathbb{N}$ with $p^n > 2^{n-1}q^n (n \in \mathbb{N}, n \geq 2, p \neq 2q)$ the $k-$tuple $(p^n - 2^{n-1}q^n, 2^{n-1}q^n, 2^{n-1}q^n, \cdots, 2^{n-1}q^n)$ is such that sum of its any two coordinates is nth power of a positive integer i.e. p^n or $(2q)^n$. Such $k-$tuples are infinitely many and first coordinate is different from the other coordinates Above are trivial examples of $k-$tuples in which sum of any two coordiantes is nth power of a positive integer.

3.2 For determination of distinct $a, b, c \in \mathbb{N}$ with $n \geq 2$, such that $a+b, a+c, b+c$ are nth power of positive integers.

We consider $a < b < c$ and $r, s, t \in \mathbb{Z} = \{0, \pm1, \pm2, \cdots\}$ such that $r < s < t$ and $a + b = r^n, a + c = s^n, b + c = t^n$.

Above equations in matrix form is

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} r^n \\ s^n \\ t^n \end{bmatrix}$$

Premultiply by $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$, the inverse of coefficient matrix $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$, we get

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} r^n \\ s^n \\ t^n \end{bmatrix}$$

which gives

$$\begin{aligned} a &= \frac{1}{2}(r^n + s^n - t^n) \\ b &= \frac{1}{2}(r^n - s^n + t^n) \\ c &= \frac{1}{2}(-r^n + s^n + t^n) \end{aligned} \tag{3.2.1}$$

It is easy to prove a, b, c are relatively prime if and only if r, s, t are relatively prime. [$\gcd(a, b, c) = d > 1 \Rightarrow \gcd(a + b, a + c, b + c) \geq d \Rightarrow \gcd(r, s, t) > 1$ etc.]

Note that $r \in \mathbb{N} \Rightarrow r, s, t \in \mathbb{N}$.

By equation (3.2.1), for $\gcd(a, b, c) = 1$; exactly one from r, s, t is even and remaining two are odd numbers.

3.3 For $n = 2$.

3.3.1 Determination of a triplet (a, b, c) of distinct positive integers such that $|a-b|, |a-c|, |b-c|$ are perfect squares.

We know all primitive Pythagorean triples are $(2st, s^2 - t^2, s^2 + t^2)$ where $s < t$ are any positive integers relatively prime with different parity [37], [4].

Consider any $k \in \mathbb{N}$. Then triples (infinitely many) $(k, (2st)^2 + k, (s^2 + t^2)^2 + k)$, $(k, (s^2 - t^2)^2 + k), ((s^2 + t^2)^2 + k)$ are such that, in any such triples, difference of any two out of three is a perfect square. For example

$(1, 10, 26), (1, 17, 26), (2, 11, 27), (2, 18, 27), \cdots$

$(1, 26, 170), (1, 145, 170), (2, 27, 171), (2, 146, 171) \cdots$

For any integer $k > max\{(2st)^2, (s^2 - t^2)^2\}$, the triples (infinitely many) $(k, k - (2st)^2, k + (s^2 - t^2)^2), (k, k + (2st)^2, k - (s^2 - t^2)^2)$ are such that, in any such triples, difference of any two out of three is a perfect square. For example

$(17, 1, 26), (17, 33, 8), (18, 2, 27), (18, 34, 9), \cdots$

$(145, 289, 120), (145, 1, 170), (146, 290, 121), (146, 2, 171), \cdots$

We consider $k > (2st)^2$ or $k > (s^2 - t^2)$ to obtain such triples. For example $(10, 26, 1), (26, 1, 170)$.

If $(a, b, c) \in \mathbb{N}$ is such that $|a - b|, |a - c|, |b - c|$ are perfect squares, then so is $(a + k, b + k, c + k)$ for all $k \in \mathbb{N}$.

3.3.2 Determination of a triplet (a, b, c) of distinct positive integers such that $a + b, a + c, b + c$ are perfect squares.

We consider $a < b < c$ i.e. $a + b < a + c < b + c$ and particular cases:

Case 1. Let $a + b = p^2, a + c = (p + 1)^2$ and $b + c = (p + 2)^2, p \in \mathbb{N}$.

Then $b - a = 2p + 3, c - a = 4p + 4$

$\Rightarrow b = a + 2p + 3, c = a + 4p + 4$ and $b + c = 2a + 6p + 7 = (p + 2)^2$

$\Rightarrow 2a = p^2 - 2p - 3$ i.e. $a = \frac{p^2-2p-3}{2} = \frac{(p-a)^2}{2} - 2$

For $p = 5, a = \frac{25-10-3}{2} = 6 \in \mathbb{N}$ and ,then $b = 6 + 10 + 3 = 19, c = 6 + 20 + 4 = 30$

$\Rightarrow (6, 19, 30)$ is a triplet for which sum of any two different coordinates is a perfect square.

For $p = 7, a = \frac{49-14-3}{2} = 16 \in \mathbb{N}$ and then $b = 16 + 14 + 3 = 33, c = 16 + 28 + 4 = 48$

$\Rightarrow (16, 33, 48)$ is a triplet for which sum of any two different coordinates is a perfect square.

Taking $p = 9$ we get $a = 30, b = 51, c = 70$

Taking $p = 11$ we get $a = 30, b = 73, c = 96$.

In general for any $n \in \mathbb{N}, n \geq 3, p = 2n - 1$ gives

$a = 2(n^2 - 2n), b = 2(n^2 - 1) + 3, c = 2(n^2 + 2n) \in \mathbb{N}$ such that

$a+b=(2n-1)^2, a+c=(2n)^2, b+c=(2n+1)^2$ are perfect squares.Such triplets (a,b,c) are infinitely many and $gcd(a,b,c)=1$ (since $gcd(a+b,a+c,b+c)=1$).

Case 2. Let $a+b=p^2, a+c=(p+1)^2$ and $b+c=(p+3)^2, p\in\mathbb{N}$.

Then $b-a=4p+8, c-a=6p+9$, i.e. $b=a+4p+8, c=a+6p+9$.

$\Rightarrow b+c=2a+10p+17=(p+3)^2$ i.e. $a=\frac{p^2-4p-8}{2}$.

For $p=6, a=2\in\mathbb{N}$ and $b=2+24+8=34, c=2+36+9=47$

For $p=8, a=12, b=52, c=69$.

For $p=10, a=26, b=74, c=95$

For $p=12, a=44, b=100, c=125$ etc.

Thus we have triplets $(2,34,47), (12,52,69), (26,74,95), (44,100,125)$ etc such that sum of any two different coordinates is a perfect square.

In general for any $n\in\mathbb{N}, n\geq 3, p=2n$ gives

$a=2(n^2-2n-2), b=2n^2+4n+4$ and $c=2n^2+8n+5$ and

$a+b=(2n)^2, a+c=(2n+1)^2, b+c=(2n+3)^2$ are perfect squares and here $gcd(a,b,c)=1$ since $gcd(a+b,a+c,b+c)=1$ etc.

Case 3. Let $a+b=p^2, a+c=(p+2)^2, b+c=(p+3)^2, p\in\mathbb{N}$.

Then $b-a=2p+5, c-a=6p+9$.

$\Rightarrow b=a+2p+5, c=a+6p+9$ and hence $2a+8p+14=b+c=(p+3)^2$

$\Rightarrow a=\frac{p^2-2p-5}{2}=\frac{(p-1)^2}{2}-3\in\mathbb{N}$ for $p=2n-1$ where $n\in\mathbb{N}$ and $n\geq 3$.

Then $a=2n^2-4n-1, b=2n^2+2, c=2n^2+8n+2$ and $a+b=(2n-1)^2$, $a+c=(2n+1)^2, b+c=(2n+2)^2$ are perfect squares for all integers $n\geq 3$ and $gcd(a,b,c)=1$. Thus $(5,20,44), (15,34,66), (29,52,92)$ etc. are triplets of positive integers in each of which sum of any two is a perfect square.

Case 4. Let $a+b=p^2, a+c=(p+1)^2, b+c=(p+4)^2, p\in\mathbb{N}$.

Then $b-a=6p+15, c-a=8p+16$

$\Rightarrow b=a+6p+15, c=a+8p+16$.

$\therefore 2a+14p+31=b+c=(p+4)^2$

Then $a=\frac{p^2-6p-15}{2}=\frac{(p-3)^2}{2}-12\in\mathbb{N}$ for $p=2n-1$ where $n\in\mathbb{N}$ and $n\geq 5$ any.

In this case $a=2n^2-8n-4, b=2n^2+4n+5, c=2n^2+8n+4$ are positive integers with $a+b=(2n-1)^2, a+c=(2n)^2, b+c=(2n+3)^2$ and $gcd(a,b,c)=1$ for all

$n \in \mathbb{N}, n \geq 5.$

For $n = 5, (a, b, c) = (6, 75, 94)$ is a triplet of positive integers such that sum of any two coordinate is a perfect square etc.

Case 5. Let $a + b = p^2, a + c = (p + 3)^2, b + c = (p + 4)^2, p \in \mathbb{N}$.

Then $b - a = 2p + 7, c - a = 8p + 16$.

$\Rightarrow b = a + 2p + 7, c = a + 8p + 16$

$\therefore 2a + 10p + 23 = b + c = (p + 4)^2$

$\Rightarrow a = \frac{p^2-2p-7}{2} = \frac{(p-1)^2}{2} - 4 \in \mathbb{N}$ for $p = 2n - 1, n \in \mathbb{N}$ and $n \geq 3$.

Then $a = 2n^2 - 4n - 2, b = 2n^2 + 3, c = 2n^2 + 12n + 6$ and $a + b = (2n - 1)^2$, $a + c = (2n + 2)^2, b + c = (2n + 3)^2$ are perfect squares for all integers $n \geq 3$ and $gcd(a, b, c) = 1$.

For $n = 3, (a, b, c) = (4, 21, 60)$,

For $n = 4, (a, b, c) = (14, 35, 86)$ etc.

Case 6. Let $a + b = p^2, a + c = (p + 2)^2, b + c = (p + 4)^2, p \in \mathbb{N}$

Then $b - a = 4p + 12, c - a = 8p + 16$

$\Rightarrow b = a + 4p + 12, c = a + 8p + 16$

$2a + 12p + 28 = b + c = (p + 4)^2$ and hence $a = \frac{p^2-4p-12}{2} = \frac{(p-2)^2}{2} - 8 \in \mathbb{N}$ for all $p = 2n, n \in \mathbb{N}$ and $n \geq 4$.

In this case $a = 2n^2 - 4n - 6, b = 2n^2 + 4n + 6, c = 2n^2 + 12n + 10$ and $a + b = (2n)^2$, $a + c = (2n + 2)^2, b + c = (2n + 4)^2$ are perfect squares and $gcd(a, b, c) = 2$.

For $n = 4, (a, b, c) = (10, 54, 90)$.

For $n = 5, (a, b, c) = (24, 76, 120)$.

In this manner taking $a + b = p^2, a + c = q^2, b + c = r^2$ where p, q, r are positive integers with $p < q < r$ and taking q, r with $q - p, r - p$ as some positive integers we obtain various triples of positive integers (a, b, c) where sum of any two coordinates is a perfect square.

Note: There are infinitely many triplets of distinct positive integers (a, b, c) such that in each of them, sum of any two coordinates is a perfect square and they are not obtained by above six cases.

For example, $(i)(a, b, c) = (18, 882, 2482)$, where $a + b = 30^2, a + c = 50^2, b + c = 58^2$.

$(ii)(a, b, c) = (130, 270, 1026)$, where $a + b = 20^2, a + c = 34^2, b + c = 36^2$ and so on.

3.3.3 Other Triplets by Using a Triplet.

If (a, b, c) is a triplet of distinct positive integers such that sum of any two numbers is a perfect squares then so is true for (n^2a, n^2b, n^2c) for each $n \in \mathbb{N}$.
For example $(6, 19, 30)$ is a triplet of positive integers such that sum of any two coordinates is a perfect square. Then $(6 \times 4, 19 \times 4, 30 \times 4), (6 \times 9, 19 \times 9, 30 \times 9), (6 \times 16, 19 \times 16, 30 \times 16)$ etc. are such triplets.
We consider now knowing a specific triplet (a, b, c) and refering process discussed in section 3.3.2, we obtain infinitely many triplets with the specific property.

Example 3.3.1. $(18, 882, 2482)$ is a triplet where $18 + 882 = 30^2$,
$18 + 2482 = 50^2 = (30 + 20)^2, 882 + 2482 = 58^2 = (30 + 28)^2$.
Let $a < b < c$ in $\mathbb{N}$ and $a + b = p^2, a + c = (p + 20)^2, b + c = (p + 28)^2, p \in \mathbb{N}$.
Then $b - a = 16p + 384, c - a = 56p + 784$.
$\Rightarrow b = a + 16p + 384, c = a + 56p + 784$
$\therefore 2a + 72p + 1168 = b + c = (p + 28)^2$
$\Rightarrow a = \frac{(p-8)^2}{2} - 224 \in \mathbb{N}$ for $p = 2n, \in \mathbb{N}$ and $n \geq 15$.
Then $a = 2n^2 - 16n - 192, b = 2n^2 + 16n + 192, c = 2n^2 + 96n + 592$
where $a + b = (2n)^2, a + c = (2n + 20)^2, b + c = (2n + 28)^2$ are perfect squares for all $n \in \mathbb{N}, (n \geq 15)$.
For $n = 15; (a, b, c) = (18, 882, 2482)$
For $n = 16; (a, b, c) = (64, 960, 2640)$
For $n = 17; (a, b, c) = (114, 1042, 2802)$.
For $n = 18, 19, 20, \cdots$ we obtain triplets of positive integers, in each sum of any two coordinates is a perfect square.

Example 3.3.2. $(130, 270, 1026)$ is a triplet where $130 + 270 = 20^2$,
$130 + 1026 = 34^2 = (20 + 14)^2, 270 + 1026 = 36^2 = (20 + 16)^2$.
Let $a < b < c$ in $\mathbb{N}$ and $a + b = p^2, a + c = (p + 14)^2, b + c = (p + 16)^2, p \in \mathbb{N}$. Then $b - a = 4p + 60, c - a = 32p + 256$

$\Rightarrow b = a + 4p + 60, c = a + 32p + 256.$

$2a + 36p + 316 = b + c = (p + 16)^2$

$\therefore a = \frac{p^2-4p-60}{2} = \frac{(p-2)^2}{2} - 32 \in \mathbb{N}$ for $p = 2n, n \in \mathbb{N}$ and $n \geq 6$.

Then $a = 2n^2 - 4n - 30, b = 2n^2 + 4n + 30, c = 2n^2 + 60n + 226$ and $a + b = (2n)^2$, $a + c = (2n + 14)^2, b + c = (2n + 16)^2$ are perfect squares for $n \in \mathbb{N}$ and $n \geq 6$.

For $n = 6; (a, b, c) = (18, 126, 658)$

For $n = 7; (a, b, c) = (40, 156, 744)$. etc.

3.3.4 Determination of Triplets from Four Tuples.

If (a, b, c, d) is a four tuple of distinct positive integers such that $a+b, a+c, a+d, b+c, b+d, c+d$, are perfect squares, then $(a, b, c), (a, b, d), (a, c, d), (b, c, d)$ are triplets of distinct positive integers such that sum of any two of integers from the triplets is a perfect square.

$(18, 882, 2482, 4743), (4190, 10210, 39074, 83426), (7070, 29794, 71330, 172706),$ $(55967, 78722, 27554, 10082), (15710, 86690, 157346, 27554)$ are four tuples of distinct positive integers such that sum of any two of them is a perfect square [3].

Then $(4190, 10210, 39074), (4190, 10210, 83426), (4190, 39074, 83426), (10210, 39074, 83426)$ etc. are triplets of distinct positive integers such that sum of any two of them is a perfect square.

Using following identity [3]

$$(m_1^2 + n_1^2)(m_2^2 + n_2^2) = (m_1m_2 + n_1n_2)^2 + (m_1n_2 - m_2n_1)^2 \qquad (3.3.1)$$

one can obtain four tuple of distinct rational numbers such that sum of any two of them is a perfect square. If in such a four tuple atleast three are positive then we obtain a triplet of distinct positive integers such that sum of any two is a perfect square. In this method we have to obtain distinct positive integers $p_1, p_2, p_3, p_4, p_5, p_6$ such that

$p_1^2 + p_2^2 = p_3^2 + p_4^2 = p_5^2 + p_6^2$ and obtain distinct numbers a, b, c, d such that

$\{a + b, a + c, a + d, b + c, b + d, c + d\} = \{p_1^2, p_2^2, p_3^2, p_4^2, p_5^2, p_6^2\}$

Let a, b, c, d be rational numbers with $a < b < c < d$ with sum of any two of them is a square of integers. Here we have

$$a + b < a + c < a + d < b + d < c + d,$$
$$a + b < a + c < b + c < b + d < c + d.$$

Let us consider $b + c < a + d$. Then we have

$$a + b < a + c < b + c < a + d < b + d < c + d \tag{3.3.2}$$

Example 3.3.3. Now $5 = 1^2 + 2^2, 10 = 1^2 + 3^2, 13 = 2^2 + 3^2$.
By equation (3.3.1), $5 \times 13 = (2 + 6)^2 + (3 - 1)^2 = (3 + 4)^2 + (6 - 2)^2$
$\Rightarrow 65 = 8^2 + 1^2 = 7^2 + 4^2$
Now $10 \times 65 = (1^2 + 3^2)(8^2 + 1^2) = (1^2 + 3^2)(7^2 + 4^2)$
By equation (3.3.1), $650 = (8 + 3)^2 + (24 - 1)^2 = (1 + 24)^2 + (8 - 3)^2$
$= (7 + 12)^2 + (21 - 4)^2 = (21 + 4)^2 + (12 - 7)^2$
$\Rightarrow 650 = 25^2 + 5^2 = 19^2 + 17^2 = 11^2 + 23^2$
Now Consider $a + b = 5^2, a + c = 11^2, b + c = 17^2, a + d = 19^2, b + d = 23^2, c + d = 25^2$ and here $a + b + c + d = 650$.
We have $c + b = 289, c - b = 96$ and hence
$2b = 193, 2c = 385, d = 23^2 - b = 432.5, a = 5^2 - b = -71.5$
Here we have $(a, b, c, d) = (-71.5, 96.5, 192.5, 432.5)$ with sum of any two of them is a perfect square $(a < b < c < d)$, then so is for (n^2a, n^2b, n^2c, n^2d) for any $n \in \mathbb{N}$.
Taking $n = 2$ we get $(-286, 386, 770, 1730)$ is a four-tuple of integers such that sum of any two of them is a perfect square.
Thus we have a triplet $(386, 770, 1730)$ such that sum of any two of them is a perfect square.

Example 3.3.4. We have, $65 = 8^2 + 1^2 = 7^2 + 4^2$ and $5^2 = 2^2 + 1^2$
$\Rightarrow 65 \times 5 = (8^2 + 1^2)(2^2 + 1^2) = (7^2 + 4^2)(2^2 + 1^2)$
i.e. $325 = 10^2 + 15^2 = 6^2 + 17^2 = 1^2 + 18^2$ by equation (3.3.1).

Let a, b, c, d be rational numbers with $a < b < c < d$ and
$a+b < a+c < b+c < a+d < b+d < c+d$ with $a+b = 1^2, a+c = 6^2, b+c = 10^2,$
$a+d = 15^2, b+d = 17^2, c+d = 18^2$
Then $b-a = 64$, and as $b+a = 1$, so $a = -31.5, b = 32.5$ and then $c = 67.5, d = 256.5$
$(4a, 4b, 4c, 4d) = (-126, 130, 270, 1026)$ is a four tuple of integers such that sum of any two of them is a perfect sqaure.
Thus we have a triplet $(130, 270, 1026)$ of distinct positive integers such that sum of any two of them is a perfect square.

Example 3.3.5 ([3])**.** $8125 = 30^2 + 85^2 = 50^2 + 75^2 = 58^2 + 69^2$.
Let a, b, c, d be numbers such that $a < b < c < d$ with
$a+b < a+c < b+c < a+d < b+d < c+d$ and $a+b = 30^2, a+c = 50^2,$
$b+c = 58^2, a+d = 69^2, b+d = 75^2, c+d = 85^2$. Solving above for positive values of a, b, c, d we get, $a = 18, b = 882, c = 2482, d = 4743$. Then $(a, b, c, d) = (18, 882, 2482, 4743)$ is a four touple of distinct positive integers such that the sum of any two of them is a perfect square.
Hence following are triplets of distinct positive integers in which sum of any two coordinates is a perfect square.
$(18, 882, 2482), (18, 882, 4743), (18, 2482, 4743), (882, 2482, 4743)$.

3.4 For n = 3.

[61] Consider $n = 3$ in equation (3.2.1), we get

$$\begin{aligned} a &= \frac{1}{2}(r^3 + s^3 - t^3) \\ b &= \frac{1}{2}(r^3 - s^3 + t^3) \\ c &= \frac{1}{2}(-r^3 + s^3 + t^3) \end{aligned} \tag{3.4.1}$$

we choose s, t in terms of r.

3.4.1 Choice $s = r + 1$ and $r + 1 < t$ where r is odd.

In this case, we choose $t = r + 2, r + 4, r + 6, \cdots$ (odd).

Case 1. Let us consider $t = r + 2$. Then by equation (3.4.1)

$$\begin{aligned} a &= \frac{1}{2}(r^3 - 3r^2 - 9r - 7) \\ b &= \frac{1}{2}(r^3 + 3r^2 + 9r + 7) \\ c &= \frac{1}{2}(r^3 + 9r^2 + 15r + 9). \end{aligned} \tag{3.4.2}$$

We consider r is odd and $r^3 > 3r^2 + 9r + 7 (r \in \mathbb{N})$, $r \geq 7$ i.e. $r = 7, 9, 11, \cdots$ and $a \in \mathbb{N}$.

For $r = 7$; equation (3.4.2) gives

$$\begin{aligned} a &= \frac{1}{2}(7^3 + 8^3 - 9^3) = \frac{1}{2}(343 + 512 - 729) = \frac{126}{2} = 63, \\ b &= \frac{1}{2}(343 - 512 + 729) = \frac{560}{2} = 280, \\ c &= \frac{1}{2}(-343 + 512 + 729) = \frac{898}{2} = 449. \end{aligned}$$

$\Rightarrow (63, 280, 449)$ is a triplet where sum of any two coordinates is a perfect cube; $63 + 280 = 343 = 7^3, 63 + 449 = 512 = 8^3, 280 + 449 = 729 = 9^3$.

For $r = 9$; equation (3.4.2) gives

$$\begin{aligned} a &= \frac{1}{2}(9^3 + 10^3 - 11^3) = \frac{1}{2}(729 + 1000 - 1331) = \frac{398}{2} = 199, \\ b &= \frac{1}{2}(729 - 1000 + 1331) = \frac{1060}{2} = 530, \\ c &= \frac{1}{2}(-729 + 1000 + 1331) = \frac{1602}{2} = 801. \end{aligned}$$

$\Rightarrow (199, 530, 801)$ is a triplet where sum of any two coordinates is a perfect cube.

For $r = 11$; equation (3.4.2) gives

$$\begin{aligned} a &= \frac{1}{2}(11^3 + 12^3 - 13^3) = \frac{1}{2}(1331 + 1728 - 2197) = \frac{862}{2} = 431, \\ b &= \frac{1}{2}(1331 - 1728 + 2197) = \frac{1800}{2} = 900, \end{aligned}$$

$$c = \frac{1}{2}(-1331 + 1728 + 2197) = \frac{2594}{2} = 1297.$$

$\Rightarrow (431, 900, 1297)$ is a triplet where sum of any two coordinates is a perfect cube. Similarly we find such triplets by taking $r = 13, 15, 17, \cdots$ which are infinitely many in counting.

Case 2. Let $s = r + 1, t = r + 4$ (r is odd), equations in (3.4.1) are

$$\begin{aligned} a &= \frac{1}{2}(r^3 - 9r^2 - 45r - 63) \\ b &= \frac{1}{2}(r^3 + 9r^2 + 45r + 63) \\ c &= \frac{1}{2}(r^3 + 15r^2 + 51r + 65). \end{aligned} \tag{3.4.3}$$

We consider r is odd and $r^3 > 9r^2 + 45r + 63 (r \in \mathbb{N})$, $\quad r \geq 13$ i.e. $r = 13, 15, 17, \cdots$ and $a \in \mathbb{N}$.

For $r = 13$; equation (3.4.3) gives

$$\begin{aligned} a &= \frac{1}{2}(13^3 + 14^3 - 17^3) = \frac{1}{2}(2197 + 2744 - 4913) = \frac{28}{2} = 14, \\ b &= \frac{1}{2}(2197 - 2744 + 4913) = \frac{4366}{2} = 2183, \\ c &= \frac{1}{2}(-2197 + 2744 + 4913) = \frac{5460}{2} = 2730. \end{aligned}$$

$\Rightarrow (14, 2183, 2730)$ is a triplet where sum of any two of its coordinates is a perfect cube.

For $r = 15$; equation (3.4.3) gives

$$\begin{aligned} a &= \frac{1}{2}(15^3 + 16^3 - 19^3) = \frac{1}{2}(3375 + 4096 - 6859) = \frac{612}{2} = 306, \\ b &= \frac{1}{2}(3375 - 4096 + 6859) = \frac{6138}{2} = 3069, \\ c &= \frac{1}{2}(-3375 + 4096 + 6859) = \frac{7580}{2} = 3790. \end{aligned}$$

$\Rightarrow (306, 3069, 3790)$ is a triplet where sum of any two of its coordinates is a perfect cube.

For $r = 17$; equation (3.4.3) gives

$$a = \frac{1}{2}(17^3 + 18^3 - 21^3) = \frac{1}{2}(4913 + 5832 - 9261) = \frac{1484}{2} = 742,$$
$$b = \frac{1}{2}(4913 - 5832 + 9261) = \frac{8342}{2} = 4171,$$
$$c = \frac{1}{2}(-4913 + 5832 + 9261) = \frac{10180}{2} = 5090.$$

$\Rightarrow (742, 4171, 5090)$ is a triplet where sum of any two of its coordinates is a perfect cube.

Similarly we can find such countably infinite triplets by taking $r = 19, 21, 23, \cdots$

3.4.2 Choice $s = r + 1$ and $r + 1 < t$ where r is even.

In this case $t = r + 3, r + 5, r + 7, \cdots$ (odd).

Case 1. Let us consider $t = r + 3$. Then by equation (3.4.1)

$$\begin{aligned} a &= \frac{1}{2}(r^3 - 6r^2 - 24r - 26) \\ b &= \frac{1}{2}(r^3 + 6r^2 + 24r + 26) \\ c &= \frac{1}{2}(r^3 + 12r^2 + 30r + 28). \end{aligned} \tag{3.4.4}$$

Consider r is even and $r^3 > 6r^2 + 24r + 26 (r \in \mathbb{N}), \quad r \geq 10$

i.e. $r = 10, 12, 14, \cdots$ and $a \in \mathbb{N}$

For $r = 10$; equation (3.4.4) gives

$$a = \frac{1}{2}(10^3 + 11^3 - 13^3) = \frac{1}{2}(1000 + 1331 - 2197) = \frac{134}{2} = 67,$$
$$b = \frac{1}{2}(1000 - 1331 + 2197) = \frac{1866}{2} = 933,$$
$$c = \frac{1}{2}(-1000 + 1331 + 2197) = \frac{2528}{2} = 1264.$$

$\Rightarrow (67, 933, 1264)$ is a triplet where sum of any two coordinates is a perfect cube.

For $r = 12$; equation (3.4.4) gives

$$\begin{aligned} a &= \frac{1}{2}(12^3 + 13^3 - 15^3) = \frac{1}{2}(1728 + 2197 - 3375) = \frac{550}{2} = 275, \\ b &= \frac{1}{2}(1728 - 2197 + 3375) = \frac{2906}{2} = 1453, \\ c &= \frac{1}{2}(-1728 + 2197 + 3375) = \frac{3844}{2} = 1922. \end{aligned}$$

$\Rightarrow (275, 1453, 1922)$ is a triplet where sum of any two coordinates is a perfect cube.

For $r = 14$; equation (3.4.4) gives

$$\begin{aligned} a &= \frac{1}{2}(14^3 + 15^3 - 17^3) = \frac{1}{2}(2744 + 3375 - 4913) = \frac{1206}{2} = 603, \\ b &= \frac{1}{2}(2744 - 3375 + 4913) = \frac{4282}{2} = 2141, \\ c &= \frac{1}{2}(-2744 + 3375 + 4913) = \frac{5544}{2} = 2772. \end{aligned}$$

$\Rightarrow (603, 2141, 2772)$ is a triplet where sum of any two coordinates is a perfect cube.

Similarly we find countably infinite triplets by taking $r = 16, 18, 20, \cdots$

Case 2. Choose $s = r + 1, t = r + 5$ (r is even) , equation (3.4.1) gives

$$\begin{aligned} a &= \frac{1}{2}(r^3 - 12r^2 - 72r - 124) \\ b &= \frac{1}{2}(r^3 + 12r^2 + 72r + 124) \\ c &= \frac{1}{2}(r^3 + 18r^2 + 78r + 126). \end{aligned} \tag{3.4.5}$$

Consider r is even and $r^3 > 12r^2 + 72r + 124 (r \in \mathbb{N}), \quad r \geq 18$

i.e. $r = 18, 20, 22, \cdots$ and $a \in \mathbb{N}$.

For $r = 18$; equation (3.4.5) gives

$$\begin{aligned} a &= \frac{1}{2}(18^3 + 19^3 - 23^3) = \frac{1}{2}(5832 + 6859 - 12167) = \frac{524}{2} = 262, \\ b &= \frac{1}{2}(5832 - 6859 + 12167) = \frac{11140}{2} = 5570, \\ c &= \frac{1}{2}(-5832 + 6859 + 12167) = \frac{13194}{2} = 6597. \end{aligned}$$

$\Rightarrow (262, 5570, 6597)$ is a triplet where sum of any two of its coordinates is a perfect cube.

For $r = 20$; equation (3.4.5) gives

$$a = \frac{1}{2}(20^3 + 21^3 - 25^3) = \frac{1}{2}(8000 + 9261 - 15625) = \frac{1636}{2} = 818,$$
$$b = \frac{1}{2}(8000 - 9261 + 15625) = \frac{14364}{2} = 7182,$$
$$c = \frac{1}{2}(-8000 + 9261 + 15625) = \frac{16886}{2} = 8443.$$

$\Rightarrow (818, 7182, 8443)$ is a triplet where sum of any two of its coordinates is a perfect cube.

For $r = 22$; equation (3.4.5) gives

$$a = \frac{1}{2}(22^3 + 23^3 - 27^3) = \frac{1}{2}(10648 + 12167 - 19683) = \frac{3132}{2} = 1566,$$
$$b = \frac{1}{2}(10648 - 12167 + 19683) = \frac{18164}{2} = 9082,$$
$$c = \frac{1}{2}(-10648 + 12167 + 19683) = \frac{21202}{2} = 10601.$$

$\Rightarrow (1566, 9082, 10601)$ is a triplet where sum of any two of its coordinates is a perfect cube.

Similarly we find countably infinite such triplets by taking $r = 24, 26, 28, \cdots$

3.4.3 Choice $s = r + 2$ and $r + 2 < t$ where r is odd.

In this case $t = r + 3, r + 5, r + 7, \cdots$ (even).

Case 1. Let us consider $t = r + 3$. Then by equation (3.4.1),

$$\begin{aligned} a &= \frac{1}{2}(r^3 - 3r^2 - 15r - 19) \\ b &= \frac{1}{2}(r^3 + 3r^2 + 15r + 19) \\ c &= \frac{1}{2}(r^3 + 15r^2 + 39r + 35). \end{aligned} \tag{3.4.6}$$

Consider r is odd and $r^3 > 3r^2 + 15r + 19 (r \in \mathbb{N})$, $\quad r \geq 7$ i.e. $r = 7, 9, 11, \cdots$ and $a \in \mathbb{N}$

For $r = 7$; equation (3.4.6) gives

$$\begin{aligned} a &= \frac{1}{2}(7^3 + 9^3 - 10^3) = \frac{1}{2}(343 + 729 - 1000) = \frac{72}{2} = 36, \\ b &= \frac{1}{2}(343 - 729 + 1000) = \frac{614}{2} = 307, \\ c &= \frac{1}{2}(-343 + 729 + 1000) = \frac{1386}{2} = 693. \end{aligned}$$

$\Rightarrow (36, 307, 693)$ is a triplet where sum of any two coordinates is a perfect cube;

For $r = 9$; equation (3.4.6) gives

$$\begin{aligned} a &= \frac{1}{2}(9^3 + 11^3 - 12^3) = \frac{1}{2}(729 + 1331 - 1728) = \frac{332}{2} = 166, \\ b &= \frac{1}{2}(729 - 1331 + 1728) = \frac{926}{2} = 463, \\ c &= \frac{1}{2}(-729 + 1331 + 1728) = \frac{2330}{2} = 1165. \end{aligned}$$

$\Rightarrow (166, 463, 1165)$ is a triplet where sum of any two coordinates is a perfect cube.

For $r = 11$; equation (3.4.6) gives

$$\begin{aligned} a &= \frac{1}{2}(11^3 + 13^3 - 14^3) = \frac{1}{2}(1331 + 2197 - 2744) = \frac{784}{2} = 392, \\ b &= \frac{1}{2}(1331 - 2197 + 2744) = \frac{1878}{2} = 939, \\ c &= \frac{1}{2}(-1331 + 2197 + 2744) = \frac{3610}{2} = 1905. \end{aligned}$$

$\Rightarrow (392, 939, 1905)$ is a triplet where sum of any two coordinates is a perfect cube.

Similarly we find countably infinite such triplets by taking $r = 13, 15, 17, \cdots$

Case 2. For $s = r + 2, t = r + 5$ (r is odd), equation (3.4.1) gives

$$\begin{aligned} a &= \frac{1}{2}(r^3 - 9r^2 - 63r - 117) \\ b &= \frac{1}{2}(r^3 + 9r^2 + 63r + 117) \\ c &= \frac{1}{2}(r^3 + 21r^2 + 87r + 133). \end{aligned} \tag{3.4.7}$$

Consider r is odd and $r^3 > 9r^2 + 63r + 117 (r \in \mathbb{N})$, $\quad r \geq 15$ i.e. $r = 15, 17, 19, \cdots$ and $a \in \mathbb{N}$.

For $r = 15$; equation (3.4.7) gives

$$\begin{aligned} a &= \frac{1}{2}(15^3 + 17^3 - 20^3) = \frac{1}{2}(3375 + 4913 - 8000) = \frac{288}{2} = 144, \\ b &= \frac{1}{2}(3375 - 4913 + 8000) = \frac{6462}{2} = 3231, \\ c &= \frac{1}{2}(-3375 + 4913 + 8000) = \frac{9538}{2} = 4769. \end{aligned}$$

$\Rightarrow (144, 3231, 4769)$ is a triplet where sum of any two of its coordinates is a perfect cube.

For $r = 17$; equation (3.4.7) gives

$$\begin{aligned} a &= \frac{1}{2}(17^3 + 19^3 - 22^3) = \frac{1}{2}(4913 + 6859 - 10648) = \frac{1124}{2} = 562, \\ b &= \frac{1}{2}(4913 - 6859 + 10648) = \frac{8702}{2} = 4351, \\ c &= \frac{1}{2}(-4913 + 6859 + 10648) = \frac{12594}{2} = 6297. \end{aligned}$$

$\Rightarrow (562, 4351, 6297)$ is a triplet where sum of any two of its coordinates is a perfect cube.

For $r = 19$; equation (3.4.7) gives

$$\begin{aligned} a &= \frac{1}{2}(19^3 + 21^3 - 24^3) = \frac{1}{2}(6859 + 9261 - 13824) = \frac{2296}{2} = 1148, \\ b &= \frac{1}{2}(6859 - 9261 + 13824) = \frac{11422}{2} = 5711, \\ c &= \frac{1}{2}(-6859 + 9261 + 13824) = \frac{16226}{2} = 8113. \end{aligned}$$

$\Rightarrow (1148, 5711, 8113)$ is a triplet where sum of any two of its coordinates is a perfect cube.

Similarly we find countably infinite such triplets by taking $r = 21, 23, 25, \cdots$.

For choice $s = r + 3$ and $r + 3 < t$ where r is even, in this case we take $t = r + 5, r + 7, r + 9, \cdots$ (odd) and obtain countably infinite triplets of relatively prime positive numbers such that sum of any two coordinates in any of such triplet is a perfect cube.

Let (a, b, c) be a triplet of distinct positive integers such that $a + b, a + c, b + c$ are cubes of positive integers. Then $(4n^3a, 4n^3b, 4n^3c)$ is a triplet of distinct positive integers such that sum of its any two coordinates is cube of a positive integer for $n = 1, 2, 3, 4, \cdots$. This gives such infinitely many triplets by use of single triplet (a, b, c).

3.5 For $n = 4$.

Using $n = 4$ in equation (3.2.1) we get

$$\begin{aligned} a &= \frac{1}{2}(r^4 + s^4 - t^4) \\ b &= \frac{1}{2}(r^4 - s^4 + t^4) \\ c &= \frac{1}{2}(-r^4 + s^4 + t^4) \end{aligned} \tag{3.5.1}$$

we choose s, t in terms of r.

3.5.1 Choice $s = r + 1$ and $r + 1 < t$ where r is odd.

In this case $t = r + 2, r + 4, r + 6, \cdots$ (odd).

Case 1. Let us consider $t = r + 2$. Then by equation (3.5.1)

$$\begin{aligned} a &= \frac{1}{2}(r^4 - 4r^3 - 18r^2 - 28r - 15) \\ b &= \frac{1}{2}(r^4 + 4r^3 + 18r^2 + 28r + 15) \\ c &= \frac{1}{2}(r^4 + 12r^3 + 30r^2 + 36r + 17). \end{aligned} \tag{3.5.2}$$

Consider r is odd and $r^4 > 4r^3 + 18r^2 + 28r + 15 (r \in \mathbb{N})$, $\quad r \geq 9$ i.e. $r = 9, 11, 13, \cdots$ and $a \in \mathbb{N}$.

For $r = 9$; equation (3.5.2) gives

$$a = \frac{1}{2}(9^4 + 10^4 - 11^4) = \frac{1}{2}[6561 + 10000 - 14641] = \frac{1902}{2} = 960,$$
$$b = \frac{1}{2}[6561 - 10000 + 14641] = \frac{11202}{2} = 5601,$$
$$c = \frac{1}{2}[-6561 + 10000 + 14641] = \frac{18080}{2} = 9040.$$

$\Rightarrow (960, 5601, 9040)$ is a triplet where sum of any two of its coordinates is a square of perfect square (fourth power of an integer):
$960 + 5601 = 6561 = 9^4, 960 + 9040 = 10^4, 5601 + 9040 = 11^4$.
For $r = 11$; equation (3.5.2) gives

$$a = \frac{1}{2}(11^4 + 12^4 - 13^4) = \frac{1}{2}[14641 + 20736 - 28561] = \frac{6816}{2} = 3408,$$
$$b = \frac{1}{2}[14641 - 20736 + 28561] = \frac{22466}{2} = 11233,$$
$$c = \frac{1}{2}[-14641 + 20736 + 28561] = \frac{34656}{2} = 17328.$$

$\Rightarrow (3408, 11233, 17328)$ is a triplet where sum of any two coordinates is a fourth power of an integer.
For $r = 13$; equation (3.5.2) gives

$$a = \frac{1}{2}(13^4 + 14^4 - 15^4) = \frac{1}{2}[28561 + 38416 - 50625] = \frac{16352}{2} = 8176,$$
$$b = \frac{1}{2}[28561 - 38416 + 50625] = \frac{40770}{2} = 20385,$$
$$c = \frac{1}{2}[-28561 + 38416 + 50625] = \frac{60480}{2} = 30240.$$

$\Rightarrow (8176, 20385, 30240)$ is a triplet where sum of any two coordinates is a fourth power of an integer.
Similarly we find such countably infinite triplets by taking $r = 15, 17, 19, \cdots$
Case 2. Let $s = r + 1, t = r + 4$ (r is odd), equation (3.5.1) gives

$$a = \frac{1}{2}(r^4 - 12r^3 - 90r^2 - 252r - 255)$$

$$b = \frac{1}{2}(r^4 + 12r^3 + 90r^2 + 252r + 255) \tag{3.5.3}$$
$$c = \frac{1}{2}(r^4 + 20r^3 + 102r^2 + 260r + 257).$$

Consider r is odd and $r^4 > 12r^3 + 90r^2 + 252r + 255 (r \in \mathbb{N})$, $r \geq 19$ i.e. $r = 19, 21, 23, \cdots$ and $a \in \mathbb{N}$.

For $r = 19$; equation (3.5.3) gives

$$a = \frac{1}{2}(19^4 + 20^4 - 23^4) = \frac{1}{2}[130321 + 160000 - 279841] = \frac{10480}{2} = 5240,$$
$$b = \frac{1}{2}[130321 - 160000 + 279841] = \frac{250162}{2} = 125081,$$
$$c = \frac{1}{2}[-130321 + 160000 + 279841] = \frac{309520}{2} = 154760.$$

$\Rightarrow$ $(5240, 125081, 154760)$ is a triplet where sum of any two of its coordinates is a square of perfect square (fourth power of an integer):

For $r = 21$; equation (3.5.3) gives

$$a = \frac{1}{2}(21^4 + 22^4 - 25^4) = \frac{1}{2}[194481 + 234256 - 390645] = \frac{38112}{2} = 19056,$$
$$b = \frac{1}{2}[194481 - 234256 + 390625] = \frac{350850}{2} = 175425,$$
$$c = \frac{1}{2}[-194481 + 234256 + 390625] = \frac{430400}{2} = 215200.$$

$\Rightarrow$ $(19056, 175425, 215200)$ is a triplet where sum of any two coordinates is a fourth power of an integer.

For $r = 23$; equation (3.5.3) gives

$$a = \frac{1}{2}(23^4 + 24^4 - 27^4) = \frac{1}{2}[279841 + 331776 - 531441] = \frac{80176}{2} = 40088,$$
$$b = \frac{1}{2}[279841 - 331776 + 531441] = \frac{479506}{2} = 239753,$$
$$c = \frac{1}{2}[-279841 + 331776 + 531441] = \frac{583376}{2} = 291688.$$

$\Rightarrow$ $(40088, 239753, 291688)$ is a triplet where sum of any two coordinates is a fourth power of an integer.

Similarly we find such countably infinite triplets by taking $r = 25, 27, 29, \cdots$

3.5.2 Choice $s = r + 1$ and $r + 1 < t$ where r is even.

In this case $t = r + 3, r + 5, r + 7, \cdots$ (odd).

Case 1. Let us consider $t = r + 3$. Then by equation (3.5.1)

$$\begin{aligned} a &= \frac{1}{2}(r^4 - 8r^3 - 48r^2 - 104r - 80) \\ b &= \frac{1}{2}(r^4 + 8r^3 + 48r^2 + 104r + 80) \\ c &= \frac{1}{2}(r^4 + 16r^3 + 60r^2 + 112r + 82). \end{aligned} \tag{3.5.4}$$

Consider r is even and $r^4 > 8r^3 + 48r^2 + 104r + 80 (r \in \mathbb{N})$, $\quad r \geq 14$

i.e. $r = 14, 16, 18, \cdots$ and $a \in \mathbb{N}$.

For $r = 14$; equation (3.5.4) gives

$$\begin{aligned} a &= \frac{1}{2}(14^4 + 15^4 - 17^4) = \frac{1}{2}[38416 + 50625 - 83521] = \frac{5520}{2} = 2760, \\ b &= \frac{1}{2}[38416 - 50625 + 83521] = \frac{71292}{2} = 35646, \\ c &= \frac{1}{2}[-38416 + 50625 + 83521] = \frac{95730}{2} = 47865. \end{aligned}$$

$\Rightarrow (2760, 35646, 47865)$ is a triplet where sum of any two of its coordinates is a square of perfect square (fourth power of an integer):

For $r = 16$; equation (3.5.4) gives

$$\begin{aligned} a &= \frac{1}{2}(16^4 + 17^4 - 19^4) = \frac{1}{2}[65536 + 83521 - 130321] = \frac{18736}{2} = 9368, \\ b &= \frac{1}{2}[65536 - 83521 + 130321] = \frac{112336}{2} = 56168, \\ c &= \frac{1}{2}[-65536 + 83521 + 130321] = \frac{148306}{2} = 74153. \end{aligned}$$

$\Rightarrow (9368, 56168, 74153)$ is a triplet where sum of any two coordinates is a fourth power of an integer.

For $r = 18$; equation (3.5.4) gives

$$\begin{aligned} a &= \frac{1}{2}(18^4 + 19^4 - 21^4) = \frac{1}{2}[104976 + 130321 - 194481] = \frac{40816}{2} = 20408, \\ b &= \frac{1}{2}[104976 - 130321 + 194481] = \frac{169136}{2} = 84568, \\ c &= \frac{1}{2}[-104976 + 130321 + 194481] = \frac{219826}{2} = 109913. \end{aligned}$$

$\Rightarrow$ $(20408, 84568, 109913)$ is a triplet where sum of any two coordinates is a fourth power of an integer.

Similarly we find such countably infinite triplets by taking $r = 20, 22, 24, \cdots$

Case 2. Let $s = r + 1, t = r + 5$ (r is even), equation (3.5.1) gives

$$\begin{aligned} a &= \frac{1}{2}(r^4 - 16r^3 - 144r^2 - 496r - 624) \\ b &= \frac{1}{2}(r^4 + 16r^3 + 144r^2 + 496r + 624) \\ c &= \frac{1}{2}(r^4 + 24r^3 + 156r^2 + 504r + 626). \end{aligned} \tag{3.5.5}$$

Consider r is even and $r^4 > 16r^3 + 144r^2 + 496r + 624 (r \in \mathbb{N}), \quad r \geq 24$

i.e. $r = 24, 26, 28, \cdots$ and $a \in \mathbb{N}$.

For $r = 24$; equation (3.5.5) gives

$$\begin{aligned} a &= \frac{1}{2}(24^4 + 25^4 - 29^4) = \frac{1}{2}[331776 + 390625 - 707281] = \frac{15120}{2} = 7560, \\ b &= \frac{1}{2}[331776 - 390625 + 707281] = \frac{648432}{2} = 324216, \\ c &= \frac{1}{2}[-331776 + 390625 + 707281] = \frac{766130}{2} = 383065. \end{aligned}$$

$\Rightarrow$ $(7560, 324216, 383065)$ is a triplet where sum of any two coordinates is a fourth power of an integer.

For $r = 26$; equation (3.5.5) gives

$$\begin{aligned} a &= \frac{1}{2}(26^4 + 27^4 - 31^4) = \frac{1}{2}[456976 + 531441 - 923521] = \frac{64896}{2} = 32448, \\ b &= \frac{1}{2}[456976 - 531441 + 923521] = \frac{849056}{2} = 424528, \end{aligned}$$

$$c = \frac{1}{2}[-456976 + 531441 + 923521] = \frac{997986}{2} = 498993.$$

$\Rightarrow (32448, 424528, 498993)$ is a triplet where sum of any two coordinates is a fourth power of an integer.

For $r = 26$; equation (3.5.5) gives

$$a = \frac{1}{2}(28^4 + 29^4 - 33^4) = \frac{1}{2}[614656 + 707281 - 1185921] = \frac{136026}{2} = 68013,$$
$$b = \frac{1}{2}[614656 - 707281 + 1185921] = \frac{993296}{2} = 496648,$$
$$c = \frac{1}{2}[-614656 + 707281 + 1185921] = \frac{1278546}{2} = 639273.$$

$\Rightarrow (68013, 496648, 639273)$ is a triplet where sum of any two coordinates is a fourth power of an integer.

Similarly we find such countably infinite triplets by taking $r = 30, 32, 34, \cdots$

For choice $s = r + 2$ and $r + 2 < t$ where r is odd, in this case we take $t = r + 3, r + 5, r + 7, \cdots$ (even) and obtain countably infinite triplets of relatively prime positive numbers such that sum of any two coordinates of any such triplet is a fourth power of an integer. In a similar way we find countably infinite triplets of such type by taking different appropriate values of s and t.

If (a, b, c) is a triplet of distinct positive integers such that $a+b, a+c, b+c$ are fourth power of integers, then so are triplets $(8n^4a, 8n^4b, 8n^4c)$ for $n = 1, 2, 3, \cdots$.

3.6 For determination of distinct $a, b, c, d \in \mathbb{Z}$, such that $a + b, a + c, b + c, a + d, b + d, c + d$ are cube of positive integers.

We consider $a, b, c, d, p, q, r, s, t, u \in \mathbb{Z}$ with $a < b < c, p < q < r$ and
$a + b = p^3, a + c = q^3, b + c = r^3, a + d = s^3, b + d = t^3, c + d = u^3.$

Above equations in matrix form is $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} p^3 \\ q^3 \\ r^3 \\ s^3 \\ t^3 \\ u^3 \end{bmatrix}$

Premultiplying by $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 1 & 0 & 0 \end{bmatrix}$,

and noting $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

we get

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} p^3 \\ q^3 \\ r^3 \\ s^3 \\ t^3 \\ u^3 \end{bmatrix}$$

which gives

$$\begin{aligned} a &= \frac{1}{2}(p^3 + q^3 - r^3) \\ b &= \frac{1}{2}(p^3 - q^3 + r^3) \\ c &= \frac{1}{2}(-p^3 + q^3 + r^3) \\ d &= \frac{1}{2}(-p^3 - q^3 + r^3 + 2s^3) \end{aligned} \tag{3.6.1}$$

It is easy to prove a, b, c are relatively prime if and only if p, q, r are relatively prime

$[\gcd(a, b, c) = v > 1 \Rightarrow \gcd(a + b, a + c, b + c) \geq v \Rightarrow \gcd(p, q, r) > 1$ etc].

Note that $p \in \mathbb{N} \Rightarrow p, q, r \in \mathbb{N}$.

By equation (3.6.1), for $\gcd(a, b, c) = 1$; exactly one from p, q, r is even and remaining two are odd numbers.

Again by equation (3.6.1); a, b, c, d are integers if and only if all integers p, q, r are even or exactly one of them is even.

In above [in equation (3.6.1) etc] the role of t^3, u^3 is invisible.

Here clearly $a + b = p^3, a + c = q^3, b + c = r^3$ and $a + d = s^3$.

Now $t^3 = b + d = -q^3 + r^3 + s^3, u^3 = c + d = -p^3 + r^3 + s^3$

$\Rightarrow s^3 + r^3 = t^3 + q^3 = u^3 + p^3$

This gives a taxicab number , expressed as a sum of two positive algebraic cubes in three distinct ways.

By Fermat's last theorem, $a + b + c + d$ never be a cube of any positive integer.

3.6.1 Main Result.

Consider a (taxicab) number which is expressed as a sum of two cubes of positive integers in three different ways. Let it be

$$s^3 + r^3 = t^3 + q^3 = u^3 + p^3 \tag{3.6.2}$$

where $p, q, r \in \mathbb{N}$ are all even or exactly one of them is even.

Let $p < q < r$ and $p^3 + q^3 > r^3$.

Take

$$\begin{aligned}
a &= \frac{1}{2}(p^3 + q^3 - r^3) \\
b &= \frac{1}{2}(p^3 - q^3 + r^3) \\
c &= \frac{1}{2}(-p^3 + q^3 + r^3) \\
d &= \frac{1}{2}(-p^3 - q^3 + r^3 + 2s^3)
\end{aligned}$$

Then $a, b, c \in \mathbb{N}$ and $a < b < c, d = s^3 - a$ and $d \in \mathbb{N}$ iff $s^3 > a$.

Clearly $a + b = p^3, a + c = q^3, b + c = r^3, a + d = s^3$ and $b + d = s^3 + r^3 - q^3 = t^3$, $c + d = s^3 + r^3 - p^3 = u^3$ by equation (3.6.2).

Hence (a, b, c, d) is a four tuple of integers where a, b, c are positive and sum of any two of its coordinates is a cube of a positive integer.

Example 3.6.1. $T_a(3) = 87539319 = 167^3 + 436^3 = 228^3 + 423^3 = 255^3 + 414^3$

Taking $p = 255, q = 423, r = 436, s = 167$; equation (3.6.1) gives

$$
\begin{aligned}
a &= \frac{1}{2}(255^3 + 423^3 - 436^3) = \frac{1}{2}(16581375 + 75686967 - 82881856) \\
&= \frac{9386486}{2} = 4693243, \\
b &= \frac{1}{2}(16581375 - 75686967 + 82881856) = \frac{23776264}{2} = 118881132, \\
c &= \frac{1}{2}(-16581375 + 75686967 + 82881856) = \frac{141987448}{2} = 70993724, \\
d &= \frac{1}{2}(-16581375 - 75686967 + 82881856 + 2 \times 167^3) = \frac{1}{2}(-938648 + 9314926) \\
&= \frac{-71560}{2} = -35780
\end{aligned}
$$

Clearly, $a + b = 4693243 + 1188813 = 16581375 = 255^3$,

$a + c = 75686967 = 423^3, a + d = 4657463 = 167^3, b + c = 82881856 = 436^3$

$b + d = 11888132 - 35780 = 11852352 = 228^3$,

$c + d = 70993724 - 35780 = 70957944 = 414^3$.

Thus $(4693243, 11888132, 70993724, -35780)$ is a four-tuple of integers such that sum of any two coordinates is a cube of positive integer.

Example 3.6.2. $15170835645 = 517^3 + 2468^3 = 709^3 + 2456^3 = 1733^3 + 2152^3$ is the smallest cubefree taxicab number with three representations.

Taking $p = 2152, q = 2456, r = 2468$ and $s = 517$, then by equation (3.6.1),

$$
\begin{aligned}
a &= \frac{1}{2}(p^3 + q^3 - r^3) = 4873961696, \\
b &= \frac{1}{2}(p^3 - q^3 + r^3) = 5092174112, \\
c &= \frac{1}{2}(-p^3 + q^3 + r^3) = 9940473120
\end{aligned}
$$

$$d = s^3 - a = -4735773283$$

where $a + b = 2152^3, a + c = 2456^3, b + c = 2468^3, a + d = 517^3, b + d = 709^3,$
$c + d = 1733^3$.

$\Rightarrow$ Four tuple of integers $(4873961696, 5092174112, 9940473120, -4735773283)$ is such that sum of any two of its coordinates is cube of a positive integer.

Example 3.6.3. Positive integer, five representations, not cubefree:
$26059452841000 = 29620^3 + 4170^3 = 28810^3 + 12900^3 = 28423^3 + 14577^3$
$= 28423^3 + 14577^3 = 24940^3 + 21930^3$
Dividing by 8, from above we get
$3257431605125 = 14810^3 + 2085^3 = 14405^3 + 6450^3 = 12470^3 + 10965^3$
Taking $p = 10965, q = 14405, r = 14810$ and $s = 2085$, we get

$$\begin{aligned} a &= \frac{1}{2}(p^3 + q^3 - r^3) = 529531610625, \\ b &= \frac{1}{2}(p^3 - q^3 + r^3) = 788803771500, \\ c &= \frac{1}{2}(-p^3 + q^3 + r^3) = 2459563869500 \\ d &= s^3 - a = -520467646500 \end{aligned}$$

$\Rightarrow a + b = 10965^3, a + c = 14405^3, b + c = 14810^3, a + d = 2085^3, b + d = 6450^3,$
$c + d = 12470^3$

$\Rightarrow (529531610625, 788803771500, 2459563869500, -520467646500)$ is a four-tuple where sum of its any two coordinates is a cube of positive integer.

Example 3.6.4. We have $143604279 = 522^3 + 111^3 = 460^3 + 359^3 = 423^3 + 408^3$
Taking $p = 408, q = 460, r = 522, s = 111$ we get by equation (3.6.1)
$a = 11508332, b = 56408980, c = 85827668, d = -10140701$.
$\Rightarrow (11508332, 56408980, 85827668, -10140701)$ is a four-tuple where sum of its any two coordinates is a cube of positive integer.

Example 3.6.5. $T_a(5)$ gives,
$48988659276962496 = 205292^3 + 342952^3 = 221424^3 + 336588^3 = 231518^3 + 331954^3$

Taking $p = 331954, q = 336588, r = 342952, s = 205292$, we get by equation (3.6.1),
$a = 171187522268991364, b = 193916369264447300, c = 20945030988258108,$
$d = -8535530906734276$
$\Rightarrow a + b = 331954^3, a + c = 336588^3, b + c = 342952^3,$
$a + d = 205292^3, b + d = 221424^3, c + d = 231518^3$
$\Rightarrow (171187522268991364, 193916369264447300, 20945030988258108,$
$-8535530906734276)$ is a four-tuple where sum of its any two coordinates is cube of a positive integer.

Example 3.6.6. We have
$1801049058342701083 = 92227^3 + 1216500^3 = 136635^3 + 1216102^3 = 341995^3 + 1207602^3$
Taking $p = 1207602, q = 1216102, r = 1216500, s = 92227$ and using equation (3.6.1), we get
$a = 879641367671452208, b = 881407757105599000, c = 918856835019401000,$
$d = -878856901453751125$
$\Rightarrow a + b = 1207602^3, a + c = 1216102^3, b + c = 1216500^3,$
$a + d = 922273^3, b + d = 136635^3, c + d = 341995^3$
$\Rightarrow (879641367671452208, 881407757105599000, 918856835019401000,$
$-878856901453751125)$ is a four-tuple where sum of its any two coordinates is cube of a positive integer.

Result 3.6.1. Existence of four-tuple of distinct positive integers such that sum of any two of its coordinates is cube of a positive integer.

Proof. $T_a(6)$ gives
$24153319581254312065344 = 16218068^3 + 27093208^3 = 17492496^3 + 26590452^3$
$= 18289922^3 + 26224366^3$
Taking $p = 16218068, q = 17492496, r = 18289922, s = 26224366$, we get by (3.6.1),

$$a = \frac{1}{2}(16218068^3 + 17492496^3 - 18289922^3) = 1749942657207366722460,$$
$$b = \frac{1}{2}(16218068^3 - 17492496^3 + 18289922^3) = 2515826512048505687972,$$

$$c = \frac{1}{2}(-16218068^3 + 17492496^3 + 18289922^3) = 3602540998645922917476,$$
$$d = 26224366^3 - a = 1628500941335251673 7436$$

Here $a + b = 16218068^3, a + c = 17492496^3, b + c = 18289922^3$
$a + d = 26224366^3, b + d = 26590452^3, c + d = 27093208^3$
$\Rightarrow$ (1749942657207366722460, 2515826512048505687972, 3602540998645922917476, 16285009413352516737436) is a four-tuple of distinct positive integers where sum of any two of its coordinates is cube of a positive integer. □

3.7 Conclusions.

We conclude that, by taking $a+b = p^2, a+c = q^2, b+c = r^2$ where p, q, r are positive integers with $p < q < r$ and taking q, r with $q - p, r - p$ as some positive integers we obtain various triples of positive integers (a, b, c) where sum of any two coordinates is a perfect square. The set of triplets

$$S = \{(a, b, c) \in \mathbb{N}^3 | a + b, a + c, b + c \text{ are perfect squares}\}$$

is an infinite set. Since $\mathbb{N}^3$ is a countable set, so such triplets are countably infinite (denumerable). The set $\{(a, b, c) \in S | gcd(a, b, c) = 1\}$ is also countably infinite.

We can determine infinitely many triplets (a, b, c) of positive integers such that $a + b, a + c, b + c$ are nth powers of positive integers for each $n = 2, 3, 4, 5, \cdots$.
For this we have to select $r, s, t \in \mathbb{N}$ in equation (3.2.1) such that all of them are even or exactly one of them is even and all right hand sides in equation (3.2.1) are positive. In particularly we determine infinitely many triplets (a, b, c) of positive integers such that $a + b, a + c, b + c$ are nth powers of consecutive integers $(n \geq 2)$. For this by equation (3.2.1)

$$a = \frac{1}{2}(r^n + (r + 1)^n - (r + 2)^n)$$
$$b = \frac{1}{2}(r^n - (r + 1)^n + (r + 2)^n)$$

$$c = \frac{1}{2}(-r^n + (r+1)^n + (r+2)^n)$$

where $r \in \mathbb{N}$ (odd) with $r^n + (r+1)^n > (r+2)^n$.

For $n = 2$, see [4].

For $n = 3, 4$ we can take $r = 9, 11, 13, \cdots$ (see sections 2, 3).

For $n = 5$ we can take $r = 11, 13, 15, \cdots$.

For example, for $r = 11$, $(19295, 141756, 229537)$ is a triplet such that sum of any two of its coordinates is a fifth power of an integer, and for $r = 25$, $(3649047, 6116578, 8232329)$ is a triplet with the same property.

For $n = 6$ we can take $r = 13, 15, 17, \cdots$.

For example, for $r = 13$, $(482860, 4343949, 65046676)$ is a triplet such that sum of any two of its coordinates is a sixth power of an integer, and

for $r = 25$, $(82817956, 161322669, 226097820)$ is a triplet with the same property.

Similarly we obtain infinitely many triplets (a, b, c) with $a + b = r^n, a + c = (r+1)^n$, $b + c = (r+3)^n$ where $r \in \mathbb{N}$ is even, $n = 5, 6, 7, \cdots$ and $r^n + (r+1)^n > (r+3)^n$.

From result 3.6.1, there is a four tuple of distinct positive integers such that sum of its any two coordinates is cube of a positive integer. If (a, b, c, d) be such a four tuple of distinct positive integers, then for any $n \in \mathbb{N}$, $(4n^3a, 4n^3b, 4n^3c, 4n^3d)$ is a four tuple of distinct positive integers where sum of any two of its coordinates is cube of a positive integer. Thus there are infinitely many such four tuples of distinct positive integers.

For any integer $n \geq 4$, if we have $p, q, r, s, t, u \in \mathbb{N}$ such that $s^n + r^n = t^n + q^n = u^n + p^n$ with $p < q < r$ and $s^n > p^n + q^n - r^n > 0$, then taking

$$a = \frac{1}{2}(p^n + q^n - r^n)$$
$$b = \frac{1}{2}(p^n - q^n + r^n)$$
$$c = \frac{1}{2}(-p^n + q^n + r^n)$$
$$d = s^n - a$$

(where all p, q, r are even or exactly one of them is even) we get a four-tuple of positive integers (a, b, c, d) such that sum of its any two coordinates is nth power of a positive integer, i.e. $a + b = p^n, a + c = q^n, b + c = r^n, a + d = s^n, b + d = t^n, c + d = u^n$.
For researchers, there are open problems for $n = 4, 5, \cdots$ to determine four-tuple (a, b, c, d) of positive integers.