# Chapter 4

# 2nd best element rationalization with general domain

## 4.1 Introduction

In the previous chapter we discussed the case of second best element rationalization of a choice function with full domain, i.e., a choice function $C : \Sigma \mapsto \Sigma$. In this chapter we relax the domain restriction of the choice function. We allow the choice function to take the form $C : D \mapsto \Sigma$, where $\emptyset \neq D \subseteq \Sigma$. We try to find a necessary and sufficient condition for a choice function with general domain to be second best element rationalizable.

## 4.2 Notation and Definitions

Let $D$ be a non-empty collection of non-empty subsets of $X$, i.e.,

$$\emptyset \neq D \subseteq 2^X - \{\emptyset\}.$$

Let a choice function $C$ be defined over $D$.

$$C : D \mapsto \Sigma \quad \text{such that} \quad C(A) \subseteq A \quad \text{for all} \quad A \in D.$$

We define a set $\lambda$ such that,

$$\lambda = \{S \in D \mid C(S) \neq S\}.$$

For every set in $\lambda$ choice set is a proper subset of the original set. So in every set in $\lambda$ there is at least one element which does not belong to the choice set. Intuitively we can say that if the choice function is 2-rationalizable then these should be the sets which have a second best element.

We now define a function, $f : \lambda \mapsto 2^X - \{\emptyset\}$
such that, for all $S \in \lambda$, $f(S) \subset S \wedge f(S) \cap C(S) = \emptyset$.

The idea behind the function $f$ is simple and intuitive. If $\lambda$ is the collection of sets with second best elements then the intended interpretation is that for any set $S$ in $\lambda$, $f(S)$ is the collection of best elements in $S$.

We define the following sets:
$A_1 = \{(x, y) \mid (\exists S \in \lambda)(x \in f(S) \wedge y \in C(S))\}$

$A_2 = \{(x, y) \mid (\exists S \in \lambda)(x \in C(S) \wedge y \in S - (C(S) \cup f(S)))\}$

$A_3 = \{(x, y) \mid (\exists S \in D)(x, y \in C(S))\}$

$A_4 = \{(x, y) \mid (\exists S \in \lambda)(x, y \in f(S))\}$

Given the intended interpretation of $f(S)$ it is clear that for any $(x, y)$ that belongs to either in $A_1$ or $A_2$, we need $x$ to be strictly preferred to $y$. Whereas, if $(x, y)$ belongs to either $A_3$ or $A_4$ then we need $x$ to be indifferent to $y$.

Let $A = A_1 \cup A_2 \cup A_3 \cup A_4$. Clearly $A$ is a binary relation over $X$. Here we introduce two conditions:
*Condition 1* : $(x, y) \in A_1 \rightarrow (y, x) \notin T(A)$
*Condition 2* : $(x, y) \in A_2 \rightarrow (y, x) \notin T(A)$

Where $T(A)$ is the transitive closure of $A$. Condition 1 and condition 2 ensure T-consistency of the binary relation $A$. Clearly with these two conditions satisfied, $P(A) = A_1 \cup A_2$.

Let, $\Delta_X = \{(x, x) | x \in X\}$.
We define a binary relation $\overline{Q}$ such that, $\overline{Q} = \Delta_X \cup T(A)$.
It can be easily verified that $\overline{Q}$ is a quasi-ordering (i.e., reflexive and transitive). Let $R$ be an ordering extension of $\overline{Q}$.

***Claim***: $\overline{Q}$ is an extension of $A$.
***Proof***: It is straight forward that $(x, y) \in A$ implies $(x, y) \in \overline{Q}$. Suppose $(x, y) \in P(A)$.
$(x, y) \in P(A) \rightarrow (x, y) \in A_1 \vee (x, y) \in A_2$
$\rightarrow (y, x) \notin T(A)$
$\rightarrow (x, y) \in P[T(A)]$
$\rightarrow (x, y) \in P(\overline{Q})$
Hence $\overline{Q}$ is an extension of $A$.

Therefore $R$ is an ordering extension of $A$.

**Axiom E**: There exists a function $f : \lambda \mapsto 2^X - \{\emptyset\}$ satisfying conditions 1 and 2.

## 4.3 Necessary and sufficient condition for a choice function to be 2-rationalizable by an ordering

**Theorem 4.1** *There exists an ordering $R$ which 2-rationalizes the choice function $C$ iff it satisfies axiom E.*

**Proof**: Suppose a choice function $C$ satisfies axiom E. If $\lambda = \emptyset$ then for all

$S \in D$ we have $C(S) = S$. In that case $R = X^2$ is a 2-rationalization.

Now let $\lambda \neq \emptyset$.

*Case 1*: Let $C(S) = S$.
$C(S) = S \rightarrow (\forall x, y \in S)(xAy)$
$\rightarrow (\forall x, y \in S)(xRy)$
$\rightarrow G_1(S, R) = S$

*Case 2*: Let $C(S) \neq S$.
By construction we have $f(S) \neq \emptyset$. Let, $x \in C(S) \wedge y \in f(S)$. Clearly, $(y, x) \in P(A)$. As $R$ is an extension of $A$ it must be $yP(R)x$. Therefore, $x \notin G_1(S, R)$.

Suppose $x \notin G_2(S, R)$. Then for some $z \in S - G_1(S, R)$ we have $zPx$.
$zPx \rightarrow \sim xRz$
$\rightarrow z \notin C(S) \wedge z \notin S - [C(S) \cup f(S)]$
$\rightarrow z \in f(S)$

$z \notin G_1(S, R)$ implies that for some $w$ in $S$ we have $wPz$. $w \notin f(S)$, as $z \in f(S)$. Also, $w \notin C(S)$ as $wPx$. Again, $wPx$ implies $w \notin S - [C(S) \cup f(S)]$. Therefore, $w \notin S$ which is a contradiction.
Therefore $x \in G_2(S, R)$.

Now we prove the converse. Let $x \in G_2(S, R)$ and suppose $x \notin C(S)$. Suppose $x \in f(S)$.
$x \in f(S) \rightarrow x \in G_1(S, R)$
$\rightarrow x \notin G_2(S, R)$
This is a contradiction. Therefore, $x \notin f(S)$.
$x \in S - [C(S) \cup f(S)]$
$\rightarrow \exists z \in f(S) \wedge \exists y \in C(S) \wedge (z, y) \in P(R) \wedge (y, x) \in P(R)$
$\rightarrow x \notin G_2(S, R)$
This is a contradiction and hence $x \in C(S)$.

Therefore $C(S) \neq S$ implies that $C(S) = G_2(S, R)$.

The necessary part of the theorem is trivial and comes straight from the intuitive interpretation that was given earlier. For every set $S$ in $\lambda$, assign $f(S)$ as the collection of best elements in $S$ and axiom E will be satisfied.