Chapter 3

2nd best element rationalization with full domain

3.1 Introduction

In this chapter we would be looking for characterization of choice functions which are second best element rationalizable (2-rationalizable). Throughout this chapter we will assume that the domain of the choice function contains all non-empty subsets of X, i.e., $C : D \mapsto \Sigma$ where $D = \Sigma$; we call it a choice function with full domain. Characterization results will be discussed for choice functions which are 2-rationalizable by orderings as well as for those which are 2-rationalizable by reflexive, connected and acyclic binary relations.

3.2 Axioms

We introduce some axioms here that we wish to use in characterization of a 2-rationalizable choice function.

A.3.1: $(\forall S \in \Sigma)[B_S \cap C(S) \neq \emptyset \rightarrow (\forall S' \subseteq S)(S' \neq \emptyset \rightarrow C(S') = S')]$

A.3.2: $(\forall S \in \Sigma)[B_S \neq S \rightarrow C(S) = \{z \in S \mid P_z \cap S \neq \emptyset \wedge P_z \cap S \subseteq B_S\}]$

A.3.3: $(\forall S \in \Sigma)(\forall x \in S)[x \notin B_S \rightarrow B_S \subseteq P_x]$

Axiom A.3.1 requires that if for any set a best element is chosen then there is no unchosen element in any of its subsets. In a 2-rationalizable choice function the only case when a best element makes its way to a choice set is when there is no second best element. Notice that in case of a 2-rationalizable choice function with full domain, the absence of a second best element ensures that all elements are best elements. So there is no reason to discriminate between alternatives while making a choice. That makes the requirement of the axiom A.3.1 intuitively clear. If however, not all elements are best then there exists a second best element. Axiom A.3.2 takes care of the situation when a second best element indeed exists. It requires that in such a situation the chosen elements are those for which a preferred element exists and any element preferred to them must be a best element. So it ensures that the chosen elements are second best. Axiom A.3.3 takes care of transitivity of the binary relation. It requires that best elements are preferred to any non-best element.

3.3 2-Rationalization

3.3.1 Necessary and sufficient condition for a choice function to be 2-rationalizable by a reflexive, connected and acyclic binary relation

Proposition 3.1 *A choice function is 2-rationalizable if it satisfies A.3.1 and A.3.2.*

Proof: Let the choice function C satisfy A.3.1 and A.3.2. It will be shown that R_2 2-rationalizes the choice function C.

Case 1: Let $G(S, R_2) = S$.
By definition, $G(S, R_2) = B_S$. Also $C(S)$ is non-empty.
Therefore, $B_S \cap C(S) \neq \emptyset$.
By A.3.1, $C(S) = S$. Hence, $G(S, R_2) = C(S)$.

Case 2: Let $G(S, R_2) \neq S$.
We shall first show that $C(S) \subseteq G(S - G(S, R_2), R_2)$. Let, $x \in C(S)$.
Suppose $x \notin G(S - G(S, R_2), R_2)$ and $x \in G(S, R_2)$.

$x \in G(S, R_2) \rightarrow B_S \cap C(S) \neq \emptyset$
By A.3.1, therefore, $(\forall S' \subseteq S)(S' \neq \emptyset \rightarrow C(S') = S')$.
$\rightarrow (\forall x, y \in S)(C(\{x, y\}) = \{x, y\})$
$\rightarrow G(S, R_2) = S$
But this is a contradiction. Therefore, $x \in S - G(S, R_2)$.

Suppose $x \notin G(S, R_2) \ \wedge \ x \notin G(S - G(S, R_2), R_2)$.

$x \notin G(S, R_2) \rightarrow x \notin B_S$
By A.3.2, therefore, $C(S) = \{z \in S \mid P_z \cap S \neq \emptyset \wedge P_z \cap S \subseteq B_S\}$.

$x \notin G(S - G(S, R_2), R_2) \ \wedge \ x \in S - G(S, R_2) \rightarrow (\exists y \in S - G(S, R_2))(y P_2 x)$
$\rightarrow y \in P_x \cap S$

$y \in S - G(S, R_2) \rightarrow y \notin G(S, R_2)$
$\rightarrow y \notin B_S$

$y \in P_x \cap S \wedge y \notin B_S \rightarrow P_x \cap S \nsubseteq B_S$
But this is a contradiction. So $x \in G(S - G(S, R_2), R_2)$ and therefore, $C(S) \subseteq G(S - G(S, R_2), R_2)$.

Now we show that $G(S - G(S, R_2), R_2) \subseteq C(S)$. Let $x \in G(S - G(S, R_2), R_2)$.

$x \in G(S - G(S, R_2), R_2) \rightarrow x \notin G(S, R_2)$
$\rightarrow x \notin B_S$
By A.3.2, therefore, $C(S) = \{z \in S \mid P_z \cap S \neq \emptyset \wedge P_z \cap S \subseteq B_S\}$.

$x \notin G(S, R_2) \rightarrow (\exists y \in S)(y P_2 x)$

$\rightarrow P_x \cap S \neq \emptyset$

Suppose $x \notin C(S)$. Then surely $P_x \cap S \not\subseteq B_S$.
$P_x \cap S \not\subseteq B_S \rightarrow (\exists z \in S)[z \in P_x \quad \wedge \quad z \notin B_S]$

$z \in P_x \rightarrow zP_2x$
$\rightarrow z \notin S - G(S, R_2) \qquad [\because \quad x \in G(S - G(S, R_2), R_2)]$
$\rightarrow z \in G(S, R_2)$
$\rightarrow z \in B_S$
But this is a contradiction. So $x \in C(S)$ and therefore $G(S - G(S, R_2), R_2) \subseteq C(S)$. Hence $C(S) = G(S - G(S, R_2), R_2)$.
From case 1 and case 2 we conclude that R_2 is a 2-rationalization of the choice function C. Hence the proposition is proved.

Proposition 3.2 *A choice function that is 2-rationalizable satisfies A.3.1 and A.3.2.*

Proof: Let R be a binary relation on X which is a second best element rationalization of the choice function C. As the choice sets are non-empty and $D = \Sigma$ it follows that R is reflexive and connected. Also R must be acyclic, otherwise there will be some set in the domain of the choice function for which we would not have any best element and hence, no second best element.

Let $(\exists S \in \Sigma)(B_S \cap C(S) \neq \emptyset)$.

Suppose, $G(S, R) \neq S$. Then by the definition of 2-rationalization and acyclicity of R we have, $C(S) = G(S - G(S, R), R)$. Consider any x in $C(S)$.
$x \in C(S)$
$\rightarrow x \in G(S - G(S, R), R)$
$\rightarrow x \notin G(S, R)$
$\rightarrow (\exists y \in S)(yPx)$

$\rightarrow C(\{x, y\}) = \{x\}$
$\rightarrow x \notin B_S$
But then $B_S \cap C(S) = \emptyset$, which is a contradiction. So, $G(S, R) = S$.
$G(S, R) = S$
$\rightarrow (\forall x, y \in S)(xRy)$
$\rightarrow (\forall S' \subseteq S)(G(S', R) = S')$
$\rightarrow (\forall S' \subseteq S)[C(S') = G(S', R) = S']$
Hence A.3.1 holds.

We now show that A.3.2 is also satisfied. Let $(\exists S \in \Sigma)(B_S \neq S)$.

Suppose $G(S, R) = S$.
$G(S, R) = S$
$\rightarrow (\forall x, y \in S)(xRy)$
$\rightarrow (\forall S' \subseteq S)(S' = G(S', R))$
$\rightarrow (\forall S' \subseteq S)[C(S') = G(S', R) = S']$
$\rightarrow (\forall x, y \in S)(C(\{x, y\}) = \{x, y\})$
$\rightarrow (\forall x \in S)(x \in B_S)$
But this violates $B_S \neq S$. So, $G(S, R) \neq S$. Then by the definition of 2-rationalization and acyclicity of R we have, $C(S) = G(S - G(S, R), R)$. Now consider any z in $C(S)$.

$z \in C(S) \rightarrow z \in G(S - G(S, R), R)$
$\rightarrow z \notin G(S, R)$
$\rightarrow (\exists y \in S)(yPz)$
$\rightarrow (\exists y \in S)[C(\{z, y\} = \{z\}]$
$\rightarrow P_z \cap S \neq \emptyset \qquad (3.1)$

Suppose $P_z \cap S \nsubseteq B_S$.
$P_z \cap S \nsubseteq B_S \rightarrow (\exists y \in S)(y \in P_z \wedge y \notin B_S)$
$y \notin B_S \rightarrow (\exists w \in S)(C(\{y, w\}) = \{y\})$
$\rightarrow wPy$
$y \in P_z \rightarrow C(\{y, z\}) = \{z\}$

$\rightarrow yPz$
$z \in G(S - G(S,R), R) \rightarrow y \in G(S,R)$
$\rightarrow \sim wPy$
This is a contradiction. Therefore, $P_z \cap S \subseteq B_S$. (3.2)

$$(3.1) \wedge (3.2) \rightarrow (\forall z \in S)(z \in C(S) \rightarrow P_z \cap S \neq \emptyset \wedge P_z \cap S \subseteq B_S) \quad (3.3)$$

Now we prove the converse. Let $z \in S$ and $P_z \cap S \neq \emptyset$ and $P_z \cap S \subseteq B_S$.
$P_z \cap S \neq \emptyset \rightarrow (\exists y \in S)[C(\{y,z\}) = \{z\}]$
$\rightarrow (\exists y \in S)(yPz)$
$\rightarrow z \notin G(S,R)$
Consider any y in S such that yPz.
$yPz \rightarrow C(\{y,z\}) = \{z\}$
$\rightarrow y \in P_z \cap S$
$\rightarrow y \in B_S$
$\rightarrow (\forall x \in S)(x \in C(\{x,y\}))$
$\rightarrow (\forall x \in S)(\sim xPy)$
$\rightarrow (\forall x \in S)(yRx)$
$\rightarrow y \in G(S,R)$
Therefore, $(\forall w \in S - G(S,R))(zRw)$.
$(\forall w \in S - G(S,R))(zRw) \rightarrow z \in G(S - G(S,R), R)$
$\rightarrow z \in C(S)$ (3.4)

Following (3.3) and (3.4) it is proved that,
$C(S) = \{z \in S \,|\, P_z \cap S \neq \emptyset \ \wedge \ P_z \cap S \subseteq B_S\}$ and therefore A.3.2 holds.

Hence the proposition is proved.

Theorem 3.1 *There exists a second best element rationalization of a choice function C if and only if C satisfies A.3.1 and A.3.2.*

Proof: Proposition 3.1 and proposition 3.2 together establish the theorem.

3.3.2 Necessary and sufficient condition for a choice function to be 2-rationalizable by an ordering

Lemma 3.1 *The binary relation R_2 is transitive if and only if the choice function C satisfies A.3.3 and $B_S \neq \emptyset$ for all $S \in \Sigma$.*

Proof: Let C satisfy A.3.3. Also let $B_S \neq \emptyset$ for all $S \in \Sigma$. Suppose R_2 is not transitive. Then there are x, y and z in X such that xR_2y , yR_2z and zP_2x. Consider the set $T = \{x, y, z\} \in \Sigma$.
$zP_2x \rightarrow \sim xR_2z$
$\rightarrow z \notin C(\{x, z\})$
$\rightarrow x \notin B_T$
Suppose $y \in B_T$. But $xR_2y \rightarrow y \in C(\{x, y\})$. Then $y \notin P_x$ and A.3.3 is violated. Therefore $y \notin B_T$.
Then it must be $z \in B_T$ as $B_T \neq \emptyset$. But $yR_2z \rightarrow z \notin P_y$. This again contradicts A.3.3.
Therefore R_2 is transitive.

Next we prove the converse. Let, R_2 be transitive.

Suppose A.3.3 is not satisfied. Then $(\exists S \in \Sigma)(\exists x \in S)(x \notin B_S \ \wedge \ B_S \not\subseteq P_x)$.
$x \notin B_S \rightarrow (\exists z \in S)(C(\{x, z\}) = \{x\})$
$\rightarrow zP_2x$
$B_S \not\subseteq P_x \rightarrow \exists y \in B_S \ \wedge \ y \notin P_x$
$\rightarrow C(\{x, y\}) = \{x, y\}$
$\rightarrow xR_2y$
$y \in B_S \rightarrow z \in C(\{y, z\})$
$\rightarrow yR_2z$
This contradicts transitivity of R_2. Therefore A.3.3 must be satisfied.

As choice sets are not empty and we have $D = \Sigma$ it therefore follows that R_2 is reflexive and connected. If R_2 is also transitive then for all T in Σ, $B_T \neq \emptyset$ as B_T is the set of best elements in T with respect to R_2.

Hence the lemma is proved.

Theorem 3.2 *There exists a second best element ordering rationalization of a choice function C if and only if C satisfies A.3.1, A.3.2 and A.3.3.*

Proof: Let C be a choice function that satisfies A.3.1, A.3.2 and A.3.3. From theorem 3.1 we know that if C satisfies A.3.1 and A.3.2 then R_2 2-rationalizes. It remains to prove that R_2 is an ordering. As the choice sets are non-empty and $D = \Sigma$ it follows that R_2 is reflexive and connected. R_2 is a 2-rationalization and $D = \Sigma$ imply that for all $T \in \Sigma$, $B_T \neq \emptyset$. With $B_T \neq \emptyset$ and A.3.3 satisfied, from lemma 3.1 we get that R_2 is transitive.

In theorem 3.1 we have already proved that A.3.1 and A.3.2 are necessary conditions. Here we only prove the necessity of A.3.3.

Let R be a second best element ordering rationalization of C. Let $x \notin B_S$ where x is an element of S and S belongs to Σ.

$x \notin B_S \rightarrow (\exists y \in S)(C(\{x, y\}) = \{x\})$

$\rightarrow yPx$

Consider any z in B_S.

$z \in B_S \rightarrow y \in C(\{y, z\})$

$\rightarrow zRy$

$zRy \wedge yPx \rightarrow zPx \qquad [\because R \text{ is transitive}]$

$zPx \rightarrow C(\{x, z\}) = \{x\}$

$\rightarrow z \in P_x$

Therefore, $B_S \subseteq P_x$ and A.3.3 holds.

3.4 Independence of the axioms

We now briefly discuss the mutual independence of the axioms used in the characterization results.

Example 3.4.1

$X = \{a, b, c\}$
$C(\{a\}) = \{a\}$, $C(\{b\}) = \{b\}$, $C(\{c\}) = \{c\}$, $C(\{a,b\}) = \{b\}$, $C(\{b,c\}) = \{c\}$, $C(\{a,c\}) = \{a,c\}$, $C(\{a,b,c\}) = \{b\}$
We have $B_X = \{a\}$ and $a \notin P_c$. Therefore A.3.3 is violated. A.3.1 and A.3.2 are satisfied.

Example 3.4.2

$X = \{a, b, c\}$
$C(\{a\}) = \{a\}$, $C(\{b\}) = \{b\}$, $C(\{c\}) = \{c\}$, $C(\{a,b\}) = \{b\}$, $C(\{b,c\}) = \{c\}$, $C(\{a,c\}) = \{c\}$, $C(\{a,b,c\}) = \{c\}$
A.3.2 is violated. A.3.1 and A.3.3 are satisfied.

Example 3.4.3

$X = \{a, b, c\}$
$C(\{a\}) = \{a\}$, $C(\{b\}) = \{b\}$, $C(\{c\}) = \{c\}$, $C(\{a,b\}) = \{a,b\}$, $C(\{b,c\}) = \{b,c\}$, $C(\{a,c\}) = \{a,c\}$, $C(\{a,b,c\}) = \{a\}$
A.3.1 is violated. A.3.2 and A.3.3 are satisfied.