Chapter 1

Introduction

In the year 1938 Paul Samuelson published his much celebrated paper which laid the foundation of revealed preference approach to the theory of consumer behavior. The approach was distinctly novel in the sense that it tried to explain the demand function from the individual's choice behavior itself and did not need to assume the existence of some ordinal utility function. Attempts were made to see whether the axiom proposed by him, now popularly known as the weak axiom of revealed preference (henceforth WARP), ensured the existence of an ordinal utility function that can generate such choices. The weak axiom failed to ensure that but succeeded in provoking a vast pool of literature in the area throughout the last century which enriched the consumer theory as well as the theory of social choice. Later Houthakker (1950) showed that a modified and stronger version of the weak axiom is necessary and sufficient for the existence of an ordinal utility function.

Uzawa (1957) and Arrow (1959) extended the framework of revealed preference to social choice theory. Instead of looking at the demand function of a consumer they became interested in choice functions in general where individuals are freely choosing a non-empty subset from every set that belongs to a class of non-empty sets. The moment we enter the theory of social choice the question that we were asking also somewhat changes. We no longer are looking for an ordinal utility function but are searching for a preference relation that can generate such a choice function. If we can find a preference relation such that the choice sets are always the sets of best elements according to the same preference relation then the choice function is said to be rationalizable. So, we may say that, rationalizability of a choice function is like approving that the choices are not nonsensical and are indeed made by some rational individual. A long list of axioms have been proposed to this end to certify the sensible nature of a choice function.

Uzawa (1957) proposed an axiom[1] which is equivalent to the strong axiom of revealed preference (henceforth SARP). It was shown there that SARP is a necessary condition for a choice function to be rationalized by an ordering. He also showed that SARP is sufficient to ensure the existence of a preference relation[2] that can generate the same choice function in the sense that the choice sets are the sets of maximal elements according to the same preference relation; and if additionally the domain of the choice function contains all non-empty finite subsets of the set of alternatives then it is the only preference relation that can do the same.

Arrow (1959) began where Uzawa (1957) ended. In the paper Arrow argued and suggested to broaden the domain of the choice function so that it contains all non-empty finite subsets of the set of alternatives (alternatively we shall refer to such a choice function as a choice function with full domain). He then examined the relationship between different axioms of rational choice. He also proposed an axiom, commonly referred to as Arrow's axiom (henceforth AA), which requires that if some elements are chosen from a set and then the set is narrowed down in a way so that some of the chosen elements are still there in the smaller set, then while making a choice from the smaller set, no previously chosen element would be unchosen and no previously unchosen element would be chosen. He showed that with full domain the three axioms SARP, AA and WARP become equivalent and they all imply the

[1]See Uzawa (1957), axiom C1.

[2]Uzawa (1957) defined a preference relation as an asymmetric and transitive binary relation.

Chernoff's axiom[3] (henceforth CA). CA (alternatively known as condition α) requires that an element if chosen from a set should also be chosen from any of its subsets that contain the element. Arrow showed that AA (alternatively SARP or WARP, as they are equivalent) is necessary and sufficient for rationalization of a choice function by an ordering if the domain of the choice function contains all non-empty finite subsets.

Charles Plott[4] formalized the idea of path independence (henceforth PI). The axiom PI requires that no matter whether we choose directly from a set or break that set into arbitrary subsets and choose from those subsets and then again choose over the choices made from those subsets, the ultimate choice set will be the same. He showed that, unlike what Arrow thought, PI is necessary, but not sufficient for rational choice. He proposed another axiom that we would refer to as the generalized Condorcet condition (henceforth GC). It requires that if an element remains unbeaten by all other elements of a set in binary choices and all such choices are possible then that element must belong to the choice set. Plott showed that PI and GC together are necessary and sufficient for rationalization of a choice function with full domain by a reflexive, connected and quasi-transitive binary relation. Blair, Bordes, Kelly and Suzumura[5] further showed that CA and GC together are necessary and sufficient for rationalization of a choice function with full domain by a reflexive, connected and acyclic binary relation.

Even as people were looking for rationalizability conditions for choice functions with full domain there were others who continued to work without imposing such domain restriction. Marcel K. Richter[6] proposed his strong congruence axiom (henceforth SCA) which requires that if an element x is revealed preferred[7] to another element y and if x and y both belong to a

[3]See Chernoff (1954).

[4]See Plott (1973).

[5]See Blair et. al. (1976).

[6]See Richter (1966).

[7]We say an alternative x is *directly revealed preferred* to another alternative y if and only if there exists a set such that both x and y belong to that set and x is chosen. If there

set and y is chosen then x must also be chosen. He showed that SCA is both necessary and sufficient for a choice function with general domain to be rationalizable by an ordering. Suzumura (1977) argued that SARP is not a legitimate formalization of Houthakker's (1950) idea. He formalized Houthakker's idea in choice-functional context and proposed an axiom known as Houthakker's axiom (henceforth HA). Suzumura showed that HA is equivalent to SCA and is necessary and sufficient for rationalization of a choice function with general domain by an ordering.

A rational individual is often described as one who arrives at a ranking of alternative states taking into account all relevant considerations. In a situation when she enjoys the freedom to choose among those alternatives, she chooses the best element according to that ranking. It is assumed that choosing the best element serves her interest best as it helps her to achieve the highest level of well-being. If the alternatives can be defined as comprehensively described states of the world and the individual concerned can indeed arrive at a ranking over such comprehensively described states then such a description possibly would not create any problem. But it may not always be possible to describe alternatives so comprehensively due to lack of information. In such a case the ranking one may observe is only a ranking over incompletely described states. A best element according to that ranking may help her achieve highest level of well-being in a particular state of the world but may not do so if the state of the world changes. In different states of the world different values assume greater importance in ranking and if we are to use one particular ranking in different states of the world then it is quite likely that we would be giving undue importance to one value (or one set of values) over another. Therefore, if we are to confine ourselves to ranking of incompletely described alternatives, which we do for all practical purposes, then 'rationality' may not necessarily call for choosing a best element. In a choice situation an individual is free to choose an alternative that helps her achieve

is a finite sequence z_1, z_2,...,z_n of alternatives such that x is directly revealed preferred to z_1, z_1 is directly revealed preferred to z_2 and so on, and z_n is directly revealed preferred to y then we say that x is *revealed preferred* to y.

the highest level of well-being. Choosing a best element does not necessarily help her achieve that because factors that represent her well-being may not be confined only to narrowly described self-interest. Altruism, abiding social norms, allegiance to certain values are some of the factors that may influence the concept of well-being. For example, ordinarily one may like to take a seat rather than remain standing while travelling in a bus but if in a crowded bus an elderly person stands in front of him, he may like to offer his seat to that person. If the ranking we are concerned about states that sitting is preferred to standing in a bus then the person here is freely choosing an alternative that is not best but achieving the highest level of well-being.

As we mentioned earlier, the purpose of rationalizability is to certify the sensible nature of a choice function. All these necessary and sufficient conditions ensure the existence of a binary relation such that, for every subset of a set its choice set is the set of its *best elements* according to that same binary relation. It is quite straight forward that if a choice function is rationalizable then choices are well-behaved. A rather problematic proposition is to say that, for choices to be well-behaved a choice function needs to be rationalizable. If a choice function is not rationalizable in the sense that the chosen elements are not the best elements then is it necessary that the choices made are inconsistent or there may still be some scheme of harmony working underneath? Until towards the end of the last century this question was either overlooked or ignored. Revealed preference approach with its intuitive appeal and superiority over the utility approach became standard and extremely influential - so much so that different axioms of this approach were taken unquestioned as *internal consistency* conditions for a choice function. The idea was that given any choice function one may check for consistency of choices with these conditions without looking at the motivation or objective of the choice. A distinct breakthrough came in 1984 with Sen's Presidential address of the Econometric Society[8] where he argued against *a priori* imposition of internal consistency conditions of choice. He argued that consistency

[8]See Sen (1993).

of choice cannot be judged in a context-independent way. In his own words, "there is no 'internal' way - internal to the choice function itself - of determining whether a particular behavior pattern is or is not consistent. The necessity of bringing in something outside choice behavior is the issue" (Sen (1993)).

Taking the cue from Sen's argument we shall be arguing here that choosing a best element is not necessary for a choice function to be consistent. However, if there is no binary relation according to which the best elements are chosen, we can immediately see that such a choice function cannot be rationalized by standard rationalizability conditions. Nevertheless, depending on the objective (which Sen calls *external reference*) someone may reasonably choose an alternative which is not best according to her preference. Standard rationalizability conditions may judge such a choice function as inconsistent, while all this time the person may consistently be choosing, say the second best elements from the sets of alternatives. Only this consistency is not apparently visible to us. Sen has argued that we can not see that consistency unless we consider the *external reference* or the objective of the chooser. *External reference* is central to the interpretation of a choice function.

Let us consider the following choices:

$$C(\{x, y\}) = \{x\}$$
$$C(\{x, y, z\}) = \{y\}$$

These choices will violate many of the standard rationalizability conditions. Clearly there is no binary relation according to which the best elements were chosen. Does that mean that the person is inconsistent in making choices? Well, that would be too hasty to conclude. Considering the fact that the person has a binary relation $zPyPx$ and is choosing the second best elements from every set we can see that at least in terms of consistency these choices are no less consistent than the case where we always choose the best elements.

The question that one may ask then is why should anybody be at all interested in choosing an inferior element while she can always choose a best element? The situation, although, prima facie may look somewhat perplexing an explanation can be given to show that such a behavior is perfectly reasonable. Earlier we have pointed out that a best element is not invariably tied up with the highest level well-being. Therefore, a person interested in achieving the highest level of well-being is free to choose an inferior alternative. To illustrate our point we start with Sen's by now classic example. A person may love cakes and her preference may satisfy nonsatiation. Nevertheless, in a party she may not be willing to pick the very largest piece from the tray as she does not want to be taken as greedy. So consistently she picks up the second largest piece of cake. She is a maximizer but her behaviour is guided by a norm: "never pick the largest slice", which she has internalised. Now let us consider the previous choice example. If x, y, and z are sizes of cakes in increasing order then her behavior is perfectly reasonable and the choices made are consistent with the norm she has internalised.

To take another example, suppose in a family there are five brothers of different ages. Their mother might have taught them that when offered pieces of cakes one should leave the bigger pieces for younger brothers. So the eldest brother feels free to choose the fifth largest piece leaving the four biggest pieces on the tray. His immediate younger brother chooses the fourth largest piece and leaves three biggest pieces for his brothers and so on. Here every brother is again consistent in his choices although their choices fail to satisfy many of the standard rationalizability conditions.

In both of the previous examples we have seen involvement of some kind of a norm. The norm acts as a constraint to the maximisation problem. It is however not necessary always to be guided by some kind of a norm or social custom to make a choice which does not involve choosing of a best element. We may do so due to purely economic reason. Consider the second-price sealed bid auction. The auctioneer chooses the second highest price from the quotations submitted to him. He does that not because of some norm or custom but because he wants to extract the maximum amount of money from the bidders.

So we see that not choosing the best element from a set is not the same as being unreasonable. A choice function that is not best element rationalizable therefore merits further investigation. Baigent and Gaertner (1996) had begun the investigation by looking for characterization of a choice function where the individual when free to choose among the alternatives, chooses the second best element if there is a uniquely best; otherwise the best elements are chosen. They have proposed five axioms[9] and showed that together they are necessary and sufficient for the existence of an ordering (which they call a non-standard rationalization) which may generate the choice function mentioned earlier. Gaertner and Xu (1999) characterized a choice behavior where the individual when free to choose among alternatives, picks up the median element from a set of alternatives. They too proposed five axioms which they showed are collectively necessary and sufficient for the existence of a linear ordering which may generate such a choice function.

In this thesis we would first like to characterize choice behavior where an individual when exercises her freedom to choose from a set of alternatives, chooses second best elements whenever available. If a second best element is not available then best elements are picked up. A binary relation according to which such choices are made would be called a second best element rationalization (alternatively we would use the term *2-rationalization*) of the choice function. We would afterward like to generalize the choice behavior where an individual when free to choose from a set of alternatives, always picks k-th best elements, where k is a positive integer. If a k-th best element is not available then $(k-1)$-th best elements are chosen; $(k-2)$-th best elements are chosen when there is no $(k-1)$-th best element, and so on. A binary relation according to which such choices are made would be called a k-th best element rationalization (alternatively called *k-rationalization*) of

[9] See section 3, Baigent and Gaertner (1996).

the choice function. We would like to find a characterization for this general choice behavior. We define a second best element in the following way. Once we remove all best elements from a set, a 2nd best element would be a best element of the reduced set. For example, if B is the set of best elements of the set S, then a 2nd best element of S is a best element of the set $S - B$. Analogously, once we remove all best elements and 2nd best elements from a set, a 3rd best element would be a best element of the reduced set; and so on. In case of k-rationalization, with $k \geq 2$, the interpretation of a best element in terms of choice sets is different from that of best element rationalization. Unlike the latter here a best element is one that does not exclude any alternative from being chosen in pairwise choices. In other words, when a choice is made out of two alternatives of which one is a best element, the selection of the other is guaranteed.

In chapter 3, we would look for a necessary and sufficient condition for 2-rationalizability of a choice function with full domain. We begin with our search for necessary conditions. Full domain ensures that there is a best element in every non-empty subset of the set of alternatives. If not, then there would be a set (which does not have a best element) in the domain for which we would end up with an empty choice set; which is not permissible. For a 2-rationalizable choice function it is clear that the chosen elements are second best whenever available. So if we indeed ever see a best element in the choice set it must be the case that there is no second best element in that set. Again, thanks to full domain, it must also be the case that every element in that set is best. Evidently for every non-empty subset of this set, their choice sets would be the entire subsets. The first axiom proposed in chapter 3 captures this idea and is a necessary condition. Axiom A.3.1 requires that whenever a best element is chosen from a set, the choice set of every non-empty subset of that set is the entire subset.

Next let us consider a set in which not all elements are best elements. Surely there are some second best elements and the choice set would be the set of these second best elements. Obviously for every second best element there must be some elements which are preferred and each of those preferred elements ought to be a best element; otherwise a second best element ceases to be a second best. The second axiom reiterates this necessary condition. Axiom A.3.2 requires that whenever a set is not comprised only of best elements its choice set would consist of all those elements for which a preferred element exists and all elements that are preferred to it are best elements.

A 2-rationalizable choice function can be divided into two parts. One consists of cases where a second best element exists in a set and the other where there is no second best element. Notice that the two necessary conditions formalized in two axioms that we mentioned earlier apply to these two parts of a 2-rationalizable choice function; one for each part. While axiom A.3.1 describes the nature of choice sets that belong to the latter part, axiom A.3.2 does the same for the former. Together they describe the entire choice function and constitute a sufficient condition for existence of a binary relation that 2-rationalizes a choice function with full domain.

So far we were talking about 2-rationalizability of a choice function by any binary relation. It turns out that to be a 2-rationalization of a choice function with full domain, a binary relation needs at least to be reflexive, connected and acyclic. When we look for an ordering (i.e. reflexive, connected and transitive) 2-rationalization, these two axioms would not suffice and we would require one more axiom. A third necessary condition that we introduce in this case, which is formalized as axiom A.3.3, merely requires that a best element is preferred to any non-best element. This axiom takes care of transitivity. Together axioms A.3.1, A.3.2 and A.3.3 also constitute a sufficient condition for existence of an ordering that 2-rationalizes a choice function with full domain.

In chapter 4, we again consider a 2-rationalizable choice function, but now we do not impose any domain restriction. Here we look for a necessary and sufficient condition for 2-rationalizability of a choice function with general domain by an ordering. An ordering 2-rationalization ensures that a set which does not have a second best element in it must be having only best elements and its choice set must be the entire set. Our search for necessary condition begins by dividing the domain of the choice function into two categories of sets - first, with sets which are chosen entirely and second, with sets whose choice sets are not equal to the entire set. We would like to focus on the latter. Surely sets in the second category are sets with second best elements and choice sets of these sets are the sets of second best elements. Therefore, in all these sets the best elements must be hidden among those unchosen elements. Now imagine that we find out those best elements. We say 'imagine' because we do not literally want to hunt them out. All we want to say is, because those elements are there, it is technically possible that we find them and put them in a group. So in every set just imagine a group of elements within the set which are best elements of that set. This allows us to divide the set in three groups of elements - best elements, chosen elements (i.e. second best elements) and residual elements (which are neither best nor second best). Notice that for any set in the second category, following relations should hold - first, a best element should be strictly preferred to a chosen element; second, a chosen element should be strictly preferred to a residual element; and third, all best elements should be indifferent. Also notice that a fourth kind of relationship must also hold for any set in the domain, that is, all chosen elements should be indifferent. These relationships among the elements give us a binary relation. This binary relation is bound to be T-consistent[10] as it most likely would be a truncated version of an ordering (that 2-rationalizes), if not an ordering itself. A necessary condition formalized in Axiom E requires the existence of a group of elements (to whom we may assign the best element status) in every set of the second category that allows us to construct a T-consistent binary relation in the above mentioned way. Axiom E is also sufficient because such a T-consistent

[10]For any binary relation R over a set of alternatives S, an ordered pair (x, y) is said to be in the *transitive closure* of R (denoted as $T(R)$) if and only if x, y and a finite sequence $z_1, z_2, ..., z_n$ of alternatives belong to S such that, xRz_1, z_1Rz_2, z_2Rz_3,...,z_nRy. A binary relation R is said to be T-consistent if and only if whenever an ordered pair (x, y) belongs to the transitive closure of R, either xRy or $\sim yRx$.

binary relation would always allow us to get an ordering extension which would 2-rationalize the choice function.

In chapter 5, we would consider a choice function where an individual when free to choose among alternatives, always picks k-th best elements, where k is a positive integer. If a k-th best element is not available in a set then $(k-1)$-th best elements are picked up. In absence of $(k-1)$-th best elements $(k-2)$-th best elements are chosen, and so on. If k takes the value 1 then we get the classic case where best elements are chosen for which the characterization results are long known to us. So the values of k that we are really concerned about are positive integers starting from 2.

Consider a choice function which is k-rationalizable. Now think of a set in its domain that has a j-th best element in it, where $j > 1$. The existence of a j-th best element implies that there must be a $(j-1)$-th best element in the set which is preferred to our j-th best element. But then again there must be a $(j-2)$-th best element which is preferred to this $(j-1)$-th best element. Following this argument we see that in any set, for the existence of a j-th best element there need to be a sequence of distinct j number of elements in the set with each of the first $j-1$ elements being preferred to the immediate next one in the sequence and the last element in the sequence, i.e., the j-th element being the j-th best element. Notice, however, that if a j-th best element does not exist in the set then there cannot be any such sequence of distinct j number of elements present in the set; if at all there is a sequence it must be of a length less than j. So the length of such a sequence tells us the number of distinct consecutive preference levels present in the set. Of course one may find more than one such sequences present in a set in which case we have to look for the longest one. If the length of the longest sequence (i.e., the number of elements required to make that sequence) is, say l, then we know there is an l-th best element present in the set but no $(l+1)$-th best element is there.

In case of a k-rationalizable choice function with $k \geq 2$ we may check that for any two elements in the set, say x and y, if x is preferred to y then $C(\{x, y\}) = \{y\}$. Therefore in any sequence of the type we mentioned earlier, in a pair wise choice among consecutive elements, the element with higher positional denomination would be chosen. So what do we get? Given a choice function with full domain we may now check for the choice sets of the doubletons and identify the longest sequence present in any set in the domain. We would call this length of the sequence as the *order of the set.* If the length of the longest sequence in a set S is, say j where j is a positive integer, then the *order* of the set S is j; and if the choice function is at all k-rationalizable for any $k \geq 2$ then we know that this set S has a j-th best element and does not have a $(j+1)$-th best element.

We would first like to find a necessary and sufficient condition for the existence of an ordering k-rationalization of a choice function with full domain. Let us consider a set in the domain of a k-rationalizable choice function, the chosen elements of which are the k-th best elements. Now pick any element from the choice set and consider all those elements in the set that are preferred to it. Clearly this set of preferred elements cannot have any k-th best element in it and therefore cannot have k distinct and consecutive preference levels in it. On the other hand the element that we earlier picked up from the choice set was a k-th best element and therefore there must be a sequence of k consecutively preferred elements of which the last and least preferred element is the k-th best element. Leaving this k-th best element all the other $k-1$ elements must be there in the set of preferred elements, which constitute $k-1$ distinct and consecutive preference levels. So the order of this set of preferred elements ought to be $k-1$. The necessary condition formalized in axiom A.5.1 requires that whenever the order of a set is k or more, its choice set would be the set of all those elements for which the preferred elements in the set constitute a set of order $k-1$.

Let us now see what happens when a k-th best element is absent in a set. We look for the $(k-1)$-th best elements and if they are also not there, we look for the $(k-2)$-th best, and so on. Notice that no matter which elements we end up choosing, in this case they are essentially the worst elements in the set. We know, if a choice function with full domain is best element rationalizable by an ordering then it is also worst element rationalizable by an ordering and the Arrow's axiom characterizes both of these cases. Therefore for any set without a k-th best element where we choose the worst elements, the Arrow's axiom should hold good. Axiom A.5.2 formalizes this necessary condition. It requires that whenever the order of a set is less than k the Arrow's axiom would be satisfied.

Axiom A.5.1 specifies the choice sets in all those cases where the order of a set is k or more. Axiom A.5.2 takes care of cases where the order of a set is less than k. It turns out that together axioms A.5.1 and A.5.2 ensure the k-rationalizability of a choice function with full domain by an ordering, whenever $k \geq 3$. Unfortunately they fail as a sufficient condition when $k = 2$. However, as we have already uncovered necessary and sufficient conditions for 2-rationalizability in chapter 3, it does not appear to be as much of a setback.

Next we would look for necessary and sufficient conditions for a choice function with full domain to be k-rationalizable by a reflexive, connected and acyclic binary relation. Very much like the previous case we would again start by dividing the domain of the choice function into two parts - sets with a k-th best element and sets without a k-th best element. Whenever a k-th best element is there in the set, it is chosen. A k-th best element entails the presence of another $k - 1$ elements in the set that along with the k-th best element form a sequence that constitute k distinct and consecutive preference levels. Of course there can be more than one such sequence of elements in the set that constitute consecutive preference levels of different lengths; all of them having the k-th best element as the last element of the sequence. But none of these sequences can be of a length greater than k.

Let us derive a binary relation R_2 in the following way - an ordered pair of alternatives (x, y) belongs to R_2 if and only if y belongs to $C(\{x, y\})$. Notice that this derived binary relation ought to be identical to the binary relation that k-rationalizes the choice function. Let the asymmetric part of R_2 be P_2. $P_2|S$ is a restriction of P_2 over a set S. $P_2|S$ is the collection of all those ordered pairs of P_2 for which both of the alternatives in a pair belong to the set S. A binary relation $T(P_2|S)$ is a transitive closure of the relation $P_2|S$.

Let us come back to those sequences that we were talking about when a k-th best element is chosen. Notice that in any of those sequences, any two consecutive alternatives, form an ordered pair and that ordered pair belongs to P_2. If the set from which the k-th best element is chosen is called, say set S, then all ordered pairs of consecutive elements in all such sequences available in set S must belong to $P_2|S$. It follows then that if we collect all the elements from all of these sequences barring the k-th best element then every element in that set would be preferred to the k-th best element according to the binary relation $T(P_2|S)$. Notice that such a set would be of order $k-1$. This is so because the maximum length that any of those sequences might achieve and at least one of them would surely achieve when the k-th best element is removed, is $k-1$. Axiom A.5.3 formalizes this necessary condition which requires that whenever a set S is of an order k or more, its choice set is constituted of elements for which a set of all preferred elements according to the binary relation $T(P_2|S)$ is of order $k-1$.

Finally consider a set S which does not have a k-th best element in it. Let the chosen elements then be j-th best, where $j < k$. Surely then in this set S a $(j+1)$-th best element would not be available and the order of the set S must be j. It follows then that the set of elements which are preferred to any j-th best element according to the binary relation $T(P_2|S)$ is of order $j-1$. This necessary condition has been formalized in axiom A.5.4. It requires that whenever a set S is of an order less than k, its choice set would consist of all those elements for which the order of the set of preferred elements according to $T(P_2|S)$ is one less than the order of the set S.

Axiom A.5.3 specifies the choice set whenever the order of a set is k or more. Axiom A.5.4 specifies the choice set whenever the order of a set is less than k. Together they constitute a necessary and sufficient condition for k-rationalization of a choice function with full domain by a reflexive, connected and acyclic binary relation where $k \geq 2$.