CHAPTER – 3

ON THE STUDY OF DERIVED SERIES OF A FOURIER SERIES AND ALLIED SERIES BY ALMOST Nörlund SUMMABILITY METHOD

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3.1 INTRODUCTION :

In 1948 Lorentz, for the first time, defined almost convergence of a sequence, Mazhar and Siddiqui (1969) proved that a convergent sequence is almost convergent and the limits are the same. The almost summability methods are defined by the idea of almost convergence of a sequence. In 1981, Qureshi defined almost Nörlund summability methods. Almost (N, p_n) method is a generalization of ordinary Nörlund summability method. After this several mathematics tried to study the Fourier series by almost Nörlund methods. In 1984 Tripathi and Lal obtained a theorem on almost Nörlund summability of conjugate series of a Fourier series. This result is recently generalised by Singh and Singh (1995). But till now no work seems to have been done so for in the direction of study of derived series of a Fourier series and derived conjugate series of a Fourier series by almost Nörlund summability method, which as known, includes as special cases, the methods of almost (C, 1) , almost (C,α), $\alpha > 0$ and Harmonic summability methods. In an attempt to make an advance study in this direction, in the present thesis, two theorems on almost Nörlund summability of derived series of a Fourier series and derived conjugate series of a Fourier series have been established under a very general condition.

3.2 DEFINITIONS AND NOTATIONS :

Let $\sum u_n$ be an infinite series with $\{s_n\}$ as the sequence of its n-th partial sums Lorentz in 1948 has given the following definition. A bounded sequence $\{s_n\}$ is said to be almost convergent to a limit s, if

(3.2.1) $$\lim_{n\to\infty} \frac{1}{n+1}\sum_{v=m}^{n+m} s_v = s \text{, uniformly w.r. to m.}$$

Let $\{p_n\}$ be a sequence of non-zero real constants and

(3.2.2) $$P_n = p_0 + p_1 + p_2 + \dots\dots + p_n\,, \quad P_{-1} = p_{-1} = 0$$

The series $\sum u_n$ or the sequence $\{s_n\}$ is said to be almost (N, p_n) summable to s [Qureshi (1981)], if

(3.2.3) $$t_{n,m} = \frac{1}{P_n}\sum_{v=0}^{n} p_{n-v}\, s_{v,m} \quad \text{tends to s, as } n \to \infty$$

uniformly with respect to m, where

(3.2.4) $$s_{v,m} = \frac{1}{v+1}\sum_{k=v}^{v+m} s_k$$

Let f(t) be a periodic function with period 2π and integral in the sense of Lebesgue over an interval $(-\pi, \pi)$. Let its Fourier series be given by

(3.2.5) $$f(t) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt)$$

$$= \frac{1}{2}a_0 + \sum_{n=1}^{\infty} A_n(t)$$

The conjugate series of Fourier series (3.2.5) is given by

(3.2.6) $$\sum_{n=1}^{\infty} (b_n \cos nt - a_n \sin nt) \equiv \sum_{n=1}^{\infty} B_n(t)$$

The series

(3.2.7) $$\sum_{n=1}^{\infty} n(b_n \cos nt - a_n \sin nt) \equiv \sum_{n=1}^{\infty} nB_n(t)$$

which is obtained by differentiating (3.2.5) term by term is called first deried series or the derived series of Fourier series of f(t). The series conjugate to (3.2.7) is

(3.2.8) $$\sum_{n=1}^{\infty} n(a_n \cos nt + b_n \sin nt) \equiv \sum_{n=1}^{\infty} nA_n(t)$$

Let $s_n(x)$ donote the sum of the n-th terms of series (3.2.6) at t = x, then in this case

(3.2.9) $$t_{n,m} = \frac{1}{P_n} \sum_{v=0}^{n} p_{n-v} s_{v,m}$$

where $$s_{v,m} = \frac{1}{v+1} \sum_{k=v}^{v+m} s_k .$$

If $\sigma_n(x)$ denotes the sum of n-th term of the series (3.2.7) at t = x , then in this case

$$t_{n,m} = \frac{1}{P_n} \sum_{v=0}^{n} p_{n-v} \sigma_{v,m}$$

We shall use following notations

(3.2.10) $$g(t) = f(x+t) - f(x-t) - 2t\, f'(x)$$

(3.2.11) $$G(t) = \int_0^t |dg(u)|$$

(3.2.12) $$h(t) = f(x+t) + f(x-t) - 2\, f'(x)$$

(3.2.13) $$X(t) = \int_0^t |dh(u)|$$

(3.2.14) $$N_{n,m}(t) = \frac{1}{2\pi P_n} \sum_{v=0}^{n} p_{n-v} \frac{\cos mt - \cos(v+m+1)t}{2(v+1)\sin^2(t/2)}$$

(3.2.15) $$\overline{N}_{n,m}(t) \frac{1}{2\pi P_n} \sum_{v=0}^{n} p_{n-v} \frac{\sin(v+m+1)\,t - \sin mt}{2(v+1)\sin^2(t/2)}$$

(3.2.16) $$H_{n+m}(t) = \frac{-1}{4\pi} \int_{1/(n+m)}^{\pi} h(t)\, \operatorname{cosec}^2(t/2)\,dt$$

3.3 MAIN THEOREM :

A quite good number of works are known on almost Nörlund summability of a Fourier series. Tripathi and Lal, for the first time in 1984, obtained an interesting result on almost (N,p_n) summability of conjugate series of a fourier series. Working in the same direction Singh and Singh (1995) generalised Tripathi and Lal's theorem under a less stringent Condition. The purpose of this thesis is to study the derived series of a Fourier series and derived conjugate series of a Fourier series by almost (N,p_n) summability method. We shall the following theorems :

THEOREM1. If

(3.3.1) $$G(t)=\int_0^t |dg(u)| = o\left[\frac{\lambda\left(\frac{1}{t}\right)p_\tau}{\{\alpha(P_\tau)\}^\delta}\right], \text{ as } t \to +0$$

(3.3.2) $$\int_{1/_{n+m}}^{1/_{(n+m)^\delta}} \frac{|dg(t)|}{t} = o(1), \quad 0<\delta<\tfrac{1}{2}, \quad \text{as } n \to \infty .$$

and

(3.3.3) $$\lambda(n+m)P_{n+m} = O\left[\{\alpha(P_{n+m})\}^\delta\right], \text{ as } n \to \infty ,$$

uniformly with respect to m, where $\lambda(t)$ and $\alpha(t)$ are function of t, such that $\lambda(t)$. $\alpha(t)$ and $\frac{t.\lambda(t)}{\alpha(t)}$ increase monotonically with t and $\{p_n\}$ is a real, non-negative, non-increasing sequence of coefficients, such that

(3.3.4) $$\sum_{v=0}^{n} \frac{p_{n-v}}{v+1} = O\left(\frac{P_n}{n}\right)$$

then the derived Fourier series (3.2.7)is almost summable (N,p_n) to $f^1(x)$ at $t=x$.

THEOREM 2. If

$$X(t) = \int_0^t dh(u) = o\left[\frac{\lambda(1/t)p_\tau}{\{\alpha(P_\tau)\}^\delta}\right], \quad \text{as } t \to +0$$

$$\int_{1/n+m}^{1/(n+m)^\delta} \frac{|dh(t)|}{t} = o(1), \quad 0 < \delta < 1/2 \;, \quad \text{as } n \to \infty \,,$$

and $$\lambda(n+m)P_{n+m} = O\left[\{\alpha(P_{n+m})\}^\delta\right], \text{ as } n \to \infty \,,$$

uniformly with respect to m, where $\lambda(t)$ and $\alpha(t)$ are functions of t, such that $\lambda(t)$, $\alpha(t)$ and $\frac{t.\lambda(t)}{\alpha(t)}$ increase monotonically with t and $\{p_n\}$ is a real, non-negative, non-increasing sequence of coefficients, such that

$$\sum_{v=0}^{n} \frac{p_{n-v}}{v+1} = O\left(\frac{P_n}{n}\right)$$

then the derived conjugate series (3.2.8) is summable (N, p_n) to the sum

$$-\frac{1}{4\pi}\int_0^\pi h(t) \;\; \text{cosec}^2(t/2)dt$$

at every point x at which integral exists.

3.4 For the proof of our theorems, we shall use the following lemmas:

LEMMA 1. For $0 < t < \frac{1}{n+m}$

$$N_{n,m}(t) = O\,(n+m)$$

PROOF OF LEMMA 1. For $0 < t < \frac{1}{n+m}$

$$|N_{n,m}(t)| = \frac{1}{2\pi P_n}\left|\sum_{v=0}^{n} p_{n-v} \frac{\cos\, mt - \cos(v+m+1)t}{2(v+1) \quad \sin^2 t/2}\right|$$

$$= \frac{1}{2\pi P_n}\left|\sum_{v=0}^{n} p_{n-v} \frac{2\sin\left(m+\frac{v+1}{2}\right)t\ \sin\left(\frac{v+1}{2}\right)t}{2(v+1)\quad \sin^2 ½}\right|$$

$$= \frac{1}{2\pi P_n}\sum_{v=0}^{n} p_{n-v}\left|\frac{(2m+v+1)\sin(½)\ (v+1)\sin(½)}{(v+1)\quad \sin^2 ½}\right|,$$

Since $|\sin nt| \le n|\sin t|$

$$= \frac{1}{2\pi P_n}\sum_{v=0}^{n} p_{n-v}(2m+v+1)$$

$$= (2m+n+1)\frac{1}{2\pi P_n}\sum_{v=0}^{n} p_{n-v}$$

$$\le (2m+2n)\frac{1}{2\pi P_n}\quad P_n = 2(m+n)\frac{1}{2\pi}$$

$$= \frac{1}{\pi}(m+n)$$

$$= O(m+n)$$

LEMMA 2. For $\frac{1}{n+m} < t < \pi$

$$\overline{N}_{n,m}(t) = O\left(\frac{1}{t}\right)$$

PROOF OF LEMMA 2.

$$\left|\overline{N}_{n,m}(t)\right| = \frac{1}{2\pi P_n}\left|\sum_{v=0}^{n} p_{n-v} \frac{\sin(v+m+1)\,t - \sin mt}{2(v+1)\quad \sin^2 ½}\right|$$

$$= \frac{1}{2\pi P_n}\left|\sum_{v=0}^{n} p_{n-v} \frac{\cos\left(m+\frac{v+1}{2}\right)t\ \sin\left(\frac{v+1}{2}\right)t}{(v+1)\quad \sin^2 ½}\right|$$

$$= \frac{1}{2\pi P_n} \sum_{\nu=0}^{n} p_{n-\nu} \frac{\left|\cos\left(m + \frac{\nu+1}{2}\right)t\right| \cdot \left|(\nu+1)\sin(\tfrac{1}{2})\right|}{(\nu+1) \quad \sin^2 \tfrac{1}{2}}$$

$$= \frac{1}{2\pi P_n} \sum_{\nu=0}^{n} p_{n-\nu} \frac{\left|\cos\left(m + \frac{\nu+1}{2}\right)t\right|}{\sin \tfrac{1}{2}}$$

$$= \frac{1}{2\pi P_n} \sum_{\nu=0}^{n} p_{n-\nu} \left(\frac{\pi}{t}\right), \qquad \text{since } \sin \frac{t}{2} \geq \frac{t}{\pi}$$

$$= \frac{1}{2tP_n} \sum_{\nu=0}^{n} p_{n-\nu}, \text{ where } \sum_{\nu=0}^{n} p_{n-\nu} = P_n$$

$$= \frac{1}{2tP_n} P_n$$

$$\left|\overline{N}_{n,m}(t)\right| = O\left(\frac{1}{t}\right)$$

LEMMA 3. For $\frac{1}{n+m} < t < \pi$

$$N_{n,m}(t) = O\left(\frac{1}{t}\right)$$

PROOF OF LEMMA 3. For $\frac{1}{n+m} < t < \pi$

$$\left|N_{n,m}(t)\right| = \frac{1}{2\pi P_n} \left|\sum_{\nu=0}^{n} p_{n-\nu} \frac{\sin\left(m + \frac{\nu+1}{2}\right)t \, \sin\left(\frac{\nu+1}{2}\right)t}{(\nu+1) \quad \sin^2 \tfrac{1}{2}}\right|$$

$$\leq \frac{1}{2\pi P_n} \sum_{\nu=0}^{n} p_{n-\nu} \frac{\left|\sin\left(m + \frac{\nu+1}{2}\right)t\right| \left|(\nu+1)\sin(\tfrac{1}{2})\right|}{(\nu+1) \quad \sin^2(\tfrac{1}{2})}$$

Expanding sin $\left(\frac{\nu+1}{2}\right)$t in powers of sin $(t/2)$.

$$\leq \frac{1}{2\pi P_n}\sum_{\nu=0}^{n} p_{n-\nu}\frac{1}{\sin\left(\frac{t}{2}\right)}$$

$$= \frac{1}{2\pi P_n}\sum_{\nu=0}^{n} p_{n-\nu}\left(\frac{\pi}{t}\right)$$

Since $\sin\theta \geq (2/\pi)\ \sin(t/2) \geq t/\pi$, by Jordon's lemma.

$$= \frac{1}{2t}\left(\frac{P_n}{n}\right)$$

$$N_{n,m}(t) = O(1/t)$$

LEMMA 4. For $\nu < n, 0 < t < \frac{1}{n+m}$

$$\left|\int_0^{1/n+m} \cot(t/2)(1-\cos \nu t)\ dh(t)\right| = o(1)\ ,\ \text{as}\ n \to \infty$$

uniformly with respect to m.

PROOF OF LEMMA 4. Since $|\sin\ nt| \leq n|\sin\ t|$

Therefore

$$\left|\int_0^{1/n+m} \cot(t/2)(1-\cos\ \nu t)\ dh(t)\right| \leq \int_0^{1/n+m} \left|\frac{\cos(t/2)}{\sin(t/2)}2\ \sin^2\left(\frac{\nu t}{2}\right)\right|\ |dh(t)|$$

$$= \int_0^{1/n+m} \frac{\cos(t/2)}{\sin(t/2)}2(\ \nu\ \sin(t/2))^2\ |dh(t)|$$

$$= \int_0^{1/n+m} \cos(t/2)2\nu^2\ \sin(t/2)|dh(t)|$$

$$= \nu^2\int_0^{1/n+m} \sin t|dh(t)|$$

$$\leq \nu^2\int_0^{1/n+m} t|dh(t)|\ ,\qquad \text{Since}\ \sin t \leq t$$

$$= \left(\frac{\nu^2}{n+m}\right)\int_0^{1/_{n+m}} |dh(t)| \ , \qquad \text{Since } t \le \frac{1}{n+m}$$

$$\le \ \nu \int_0^{1/_{n+m}} |dh(t)|$$

$$= \nu \ o\left[\frac{\lambda(n+m)\ p_{(n+m)}}{\{\alpha(P_{(n+m)})\}^{\delta}}\right]$$

$$= \nu \left[\frac{\lambda(n+m)\cdot(n+m)\ p_{(n+m)}}{(n+m)\{\alpha(P_{(n+m)})\}^{\delta}}\right]$$

$$= o\left[\frac{\nu}{(n+m)}\right]$$

$$= o(1) \quad , \text{ as } n \to \infty,$$

uniformly with respect to m.

LEMMA 5. For $\nu < n$

$$-\frac{1}{2\pi}\int_{1/_{n+m}}^{\pi} \cot(t/2)(1-\cos \nu t)\ dh(t)$$

$$= \frac{1}{2\pi}\int_{1/_{n+m}}^{\pi} \cot(t/2)\cos \nu t\ dh(t) + H_{n+m}(x) + o(1). \quad \text{as } n \to \infty,$$

uniformly with respect to m.

PROOF OF LEMMA 5. We have

$$-\frac{1}{2\pi}\int_{1/_{n+m}}^{\pi} \cot(t/2)(1-\cos \nu t)\ dh(t)$$

$$= \frac{-1}{2\pi}\int_{1/_{n+m}}^{\pi} \cot(t/2)\ dh(t) + \frac{1}{2\pi}\int_{1/_{n+m}}^{\pi} \cot t/2\ \cos \nu t\ dh(t)$$

$$= \frac{-1}{2\pi}\left[\cot(t/2)\ (H(t))_{1/n+m}^{\pi}\right] + \frac{1}{2\pi}\int_{1/n+m}^{\pi} \frac{1}{2}\operatorname{cosec}^2(t/2)\ H(t)d(t)$$

$$+ \frac{1}{2\pi}\int_{1/n+m}^{\pi} \cot(t/2)\ \cos vt \quad dh(t)$$

$$= \frac{1}{2\pi}\frac{\cos\left(\frac{1}{2(n+m)}\right)\{\alpha(P_{n+m})\}^{\delta}}{\sin\left(\frac{1}{2(n+m)}\right)2(n+m)P_{n+m}} + H_{n+m}(x)$$

$$+ \frac{1}{2\pi}\int_{1/n+m}^{\pi} \cot(t/2)\ \cos vt \quad dh(t)$$

$$= \frac{1}{2\pi}\cos\left(\frac{1}{2(n+m)}\right)\left(\frac{\frac{1}{2(n+m)}}{\sin\left(\frac{1}{2(n+m)}\right)}\right)\left(\frac{\{\alpha(P_{n+m})\}^{\delta}}{P_{n+m}}\right) + H_{n+m}(x)$$

$$+ \frac{1}{2\pi}\int_{1/n+m}^{\pi} \cot(t/2)\ \cos vt\ dt(t)$$

$$= \frac{1}{2\pi}\cos\left(\frac{1}{2(n+m)}\right)\left(\frac{\frac{1}{2(n+m)}}{\sin\left(\frac{1}{2(n+m)}\right)}\right)(\lambda_{n+m}) + H_{n+m}(x)$$

$$+ \frac{1}{2\pi}\int_{1/n+m}^{\pi} \cot(t/2)\ \cos vt\ dh(t), \text{by } (3.3.3)$$

$$= \frac{1}{2\pi}\int_{1/n+m}^{\pi} \cot(t/2)\cos vt\ dh(t) + H_{n+m}(x) + o(1), \quad \text{as } n \to \infty,$$

uniformly with respect to m.

3.5 PROOF OF THE THEOREM 1. Denoting by $s_n(x)$ the sum of the first n-terms of the series (3.2.7) at $t = x$, we get

$$s_n(x) = \frac{1}{2\pi}\int_0^{2\pi}\left\{\frac{d}{du}\frac{\sin\left(n+\frac{1}{2}\right)(x-u)}{\sin\frac{1}{2}(x-u)}\right\}f(u)\,du$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(u)\left\{\frac{d}{du}\frac{\sin\left(n+\frac{1}{2}\right)(x-u)}{\sin\frac{1}{2}(x-u)}\right\}du$$

$$(3.5.1) \qquad = \frac{-1}{2\pi}\int_0^{\pi}\{f(n+t)-f(x-t)\}\left\{\frac{d}{dt}\frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}\right\}dt$$

put $x-u=t$ or $x-t=u$ and using definite integral property.

Now integrating by parts right hand side of eq'n (3.5.1), we get

$$s_n(x) = \frac{1}{2\pi}\int_0^{\pi}\frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}\,d\{f(x+t)-f(x-t)\}$$

$$= \frac{1}{2\pi}\int_0^{\pi}\frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}\,dg(t)+f^{1}(x)$$

Therefore,

$$s_n(x)-f^{1}(x) = \frac{1}{2\pi}\int_0^{\pi}\frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}\,dg(t)$$

So that,

$$s_{v,m}-f^{1}(x) = \frac{1}{(v+1)}\sum_{n=v}^{v+m}\{s_n(x)-f^{1}(x)\}, \quad \text{since} \quad s_{v,m} = \frac{1}{v+1}\sum_{k=v}^{v-m} s_k$$

$$= \frac{1}{2\pi(\nu+1)} \sum_{n=\nu}^{\nu+m} \left(\int_0^{\pi} \frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{1}{2}t} \, dg\,(t) \right)$$

$$s_{\nu,m} - f^{1}(x) = \frac{1}{2\pi(\nu+1)} \int_0^{\pi} \frac{\cos mt - \cos(\nu+m+1)t}{2\sin^2(\tfrac{1}{2})} \, dg(t)$$

Now,

$$\frac{1}{P_n} \sum_{\nu=0}^{n} p_{n-\nu} \{ s_{\nu,m} - f^{1}(x) \} = \frac{1}{P_n} \sum_{\nu=0}^{n} p_{n-\nu} \left\{ \frac{1}{2\pi(\nu+1)} \int_0^{\pi} \frac{\cos mt - \cos(\nu+m+1)t}{2\sin^2(\tfrac{1}{2})} dg(t) \right\}$$

$$t_{n,m} - f^{1}(x) = \int_0^{\pi} \frac{1}{2\pi P_n} \sum_{\nu=0}^{n} p_{n-\nu} \frac{\cos mt - \cos(\nu+m+1)t}{2(\nu+1)\sin^2(\tfrac{1}{2})}$$

$$= \int_0^{\pi} N_{n,m}(t) \, dg(t) \,, \qquad \text{by eq'n (3.2.14)}$$

In order to prove the theorem 1, we have to show that under our assumptions

$$\int_0^{\pi} N_{n,m}(t) dg(t) = o(1) \quad , \quad \text{as} \quad n \to \infty \,,$$

uniformly with respect to m.

Now we have

$$\int_0^{\pi} N_{n,m}(t) dg(t) = \left(\int_0^{1/_{n+m}} + \int_{1/_{n+m}}^{1/_{(n+m)^{\delta}}} + \int_{1/_{(n+m)^{\delta}}}^{\pi} \right) N_{n,m}(t) dg(t)$$

(3.5.2) $\qquad = L_1 + L_2 + L_3 \,, \quad$ say.

Now for $\quad 0 < t \le \frac{1}{n+m}$

$$N_{n,m}(t) = O(n+m) \quad , \qquad \text{by Lemma 1}$$

Hence

$$|L_1| = O\left(\int_0^{1/_{n+m}} |N_{n,m}(t)| |dg(t)| \right)$$

$$= O(n+m)\left(\int_0^{1/n+m} |dg(t)|\right)$$

$$= O\left((n+m)G\left(\frac{1}{n+m}\right)\right)$$

$$= O(n+m)\ o\left[\frac{\lambda(n+m)p_{n+m}}{\{\alpha(P_{n+m})\}^{\delta}}\right]$$

(3.5.3) $= o(1), \quad \text{as } n \to \infty$,

uniformly with respect to m.

Next, considering L_2 , we have by lemma 3.

$$|L_2| = O\left(\int_{1/(n+m)}^{1/(n+m)^{\delta}} \frac{|dg(t)|}{t}\right)$$

(3.5.4) $= o(1), \quad \text{as } n \to \infty$, by (3.3.2)

uniformly with respect to m.

Lastly, we have $|L_3| = \int_{1/(n+m)^{\delta}}^{\pi} dg(t) \frac{1}{2\pi P_n} \sum_{v=0}^{n} p_{n-v} \frac{\cos mt - \cos(v+m+1)t}{2(v+1)\sin^2(t/2)}$

$$\leq \frac{1}{2\pi P_n} \int_{1/(n+m)^{\delta}}^{\pi} \frac{p_{n-v}}{2(v+1)} \frac{\cos mt}{\sin^2(t/2)} dg(t) +$$

$$+ \frac{1}{2\pi P_n} \int_{1/(n+m)^{\delta}}^{\pi} \sum_{v=0}^{n} \frac{p_{n-v}}{2(v+1)} \frac{\cos(v+m+1)t}{\sin^2(t/2)} dg(t)$$

$$= L_{3.1} + L_{3.2} \quad \text{, say.}$$

Now using second mean value theorem, we have

$$L_{3.1} \leq \frac{1}{2\pi P_n} \sum_{v=0}^{n} \frac{p_{n-v}}{2(v+1)} \frac{1}{2\sin^2\left(\frac{1}{2(n+m)^{\delta}}\right)} \int_{1/(n+m)^{\delta}}^{\epsilon} \cos mt\, dg(t)$$

where $\frac{1}{(n+m)^{\delta}} < \in < \pi$

$$= O\left(\frac{1}{n}\right)(n+m)^{2\delta}\int_{1/(n+m)^{\delta}}^{\in} |dg(t)|$$

(3.5.5) $= o(1)$, as $n \to \infty$,

uniformly with respect to m.

Next

$$|L_{32}| = \left|\frac{1}{2\pi P_n}\int_{1/(n+m)^{\delta}}^{\pi} \sum_{v=0}^{n} \frac{p_{n-v}}{2(v+1)} \frac{\cos(v+m+1)t}{\sin^2(t/2)} dg(t)\right|$$

$$\leq \frac{1}{2\pi P_n}\int_{1/(n+m)^{\delta}}^{\epsilon_1} \sum_{v=0}^{n} \left(\frac{p_{n-v}}{2(v+1)}\right)\frac{1}{\sin^2(t/2)} dg(t)$$

using second Mean value theorem

$$= O\left(\frac{1}{n}\right)\frac{1}{\sin\left(\frac{1}{2(n+m)^{\delta}}\right)}\int_{1/(n+m)^{\delta}}^{\in_1} |dg(t)|$$

where $\left(\frac{1}{(n+m)^{\delta}}\right) < \in_1 < \pi$

$$= O\left(\frac{(n+m)^{2\delta}}{n}\right)\int_{1/(n+m)^{\delta}}^{\in_1} |dg(t)|$$

(3.5.6) $= o(1)$, as $n \to \infty$,

uniformly with respect to m.

Combining (3.5.5) and (3.5.6), we get

(3.5.7) $L_3 = o(1)$, as $n \to \infty$,

uniformly w.r. to m.

Collecting (3.5.3.), (3.5.4) and (3.5.7.), we get

$$\int_0^{\pi} N_{n,m}(t)\ dg(t) = o(1)\ ,\ \text{as}\ \ n \to \infty\ ,$$

uniformly with respect to m.

This complete the proof of the theorem 1.

PROOF OT THE THEOREM 2. Denoting $\sigma_v(x)$ the sum of the v-th terms, it the series

$$\sum_{n=1}^{\infty} n\left(b_n \cos nt - a_n \sin nt\right) = \sum_{n=1}^{\infty} n\ B_n(t),\ \text{at}\ t = x$$

we have

$$\sigma_v(x) = \frac{-1}{2\pi}\int_{-\pi}^{\pi} f(x)\ \frac{\partial}{\partial n}\left(\sum_{n=1}^{\infty} \sin(n-x)\ dn\right)$$

$$= \frac{-1}{2\pi}\int_0^{\pi} \frac{d}{dt}\left(\frac{\cos\left(\frac{t}{2}\right) - \cos\left(n - \frac{1}{2}\right)t}{\sin(t/2)}\right)\left[f(x+t) + f(x-t)\right]dt$$

$$= \frac{-1}{2\pi}\int_0^{\pi} \cot(t/2)(1-\cos vt)dh(t) - \frac{1}{2\pi}\int_0^{\pi} \cot(t/2)\cos vt\ dh(t)$$

$$+ \frac{1}{2\pi}\int_0^{\pi} \frac{\cos\left(v + \frac{1}{2}\right)t}{\sin(t/2)} dh(t)$$

$$= \frac{-1}{2\pi}\int_0^{\pi} \cot(t/2)(1-\cos vt)dh(t) + \frac{1}{2\pi}\int_0^{\pi} \frac{\cos\left(v + \frac{1}{2}\right)t - \cot(t/2)\cos vt}{\sin(t/2)} dh(t)$$

$$= \frac{-1}{2\pi}\int_0^{\pi} \cot(t/2)(1-\cos vt)dh(t) - \frac{1}{2\pi}\int_0^{\pi} \sin vt\ dh(t)$$

Hence,

$$\sigma_v(x) = \frac{-1}{2\pi}\int_0^{\pi} \left[\cot(t/2)(1-\cos vt) + \sin vt\right] dh(t)$$

$$= \frac{-1}{2\pi}\left\{\int_0^{1/_{n+m}} + \int_{1/_{n+m}}^{\pi}\right\}\left[\cot(t/2)(1-\cos \nu t)\right] dh(t) - \frac{1}{2\pi}\int_0^{\pi} \sin(\nu t)\, dh(t)$$

$$= \frac{1}{2\pi}\int_{1/_{n+m}}^{\pi} \left[\cot(t/2)\cos(\nu t)\right] dh(t)$$

$$- \frac{1}{2\pi}\int_0^{\pi} \sin(\nu t)\, dh(t) + H_{n+m}(x) + o(1), \text{ by lemma 3 and 4}$$

$$= \frac{1}{2\pi}\int_{1/_{n+m}}^{\pi} \left[\cot(t/2)\cos(\nu t)\right] dh(t) - \frac{1}{2\pi}\int_{1/_{n+m}}^{\pi} \sin(\nu t)\, dh(t)$$

$$- \frac{1}{2\pi}\int_0^{1/_{n+m}} \sin(\nu t)\, dh(t) + H_{n+m}(t) + o(1)$$

$$= \frac{1}{2\pi}\int_{1/_{n+m}}^{\pi} \frac{\cos\left(\nu + \frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)}\, dh(t) + H_{n+m}(t) + o(1)$$

Therefore,

$$\sigma_{\nu}(x) - H_{n+m}(x) = \frac{1}{2\pi}\int_{1/_{n+m}}^{\pi} \frac{\cos\left(\nu + \frac{1}{2}\right)t}{\sin(t/2)}\, dh(t) + o(1)$$

so that

$$\sigma_{\nu,m}(x) - H_{n+m}(x) = \frac{1}{\nu+1}\sum_{k=\nu}^{\nu+m} \left\{\sigma_k(x) - H_{n+m}(x)\right\}$$

$$= \frac{1}{2\pi(\nu+1)}\int_{1/_{n+m}}^{\pi} \sum_{k=\nu}^{\nu+m} \frac{\cos\left(\nu + \frac{1}{2}\right)t}{\sin(t/2)}\, dh(t) + o(1)$$

$$= \frac{1}{2\pi(\nu+1)}\int_{1/_{n+m}}^{\pi} \frac{\sin(\nu+m+1)t - \sin mt}{2\sin^2(t/2)}\, dh(t) + o(1)$$

Now

$$t_{n,m} - H_{n+m}(x) = \frac{1}{2\pi P_n}\int_{1/_{n+m}}^{\pi} p_{n-\nu}\, \frac{\sin(\nu+m+1)t - \sin mt}{2(\nu+1)\sin^2(t/2)}\, dh(t) + o(1)$$

$$= \int_{1/_{n+m}}^{\pi} \frac{1}{2\pi P_n} \sum_{v=0}^{n} p_{n-v} \frac{\sin(v+m+1)t - \sin mt}{2(v+1)\sin^2(t/2)} dh(t) + o(1)$$

$$= \int_{1/_{n+m}}^{\pi} \overline{N}_{n,m}(t)\, dh(t) + o(1) \quad , \qquad \text{by (3.2.15)}$$

we have to show that our assumptions ,

$$= \int_{1/_{n+m}}^{\pi} \overline{N}_{n,m}(t)\, dh(t) = o(1) , \qquad \text{as } n \to \infty ,$$

uniformly with respect to m.

we write

$$\int_{1/_{n+m}}^{\pi} \overline{N}_{n,m}(t)\, dh(t) = \left\{ \int_{1/_{n+m}}^{1/_{(n+m)^\delta}} + \int_{1/_{(n+m)^\delta}}^{\pi} \right\} \overline{N}_{n,m}(t)\, dh(t) + o(1)$$

(3.5.8) $$= Q_1 + Q_2 + o(1) \text{ , say}$$

Now, $$Q_1 = \int_{1/_{(n+m)}}^{1/_{(n+m)^\delta}} \overline{N}_{n,m}(t)\, dh(t)$$

$$|Q_1| = \int_{1/_{n+m}}^{1/_{n+\delta}} |\overline{N}_{n,m}(t)|\, |dh(t)|$$

$$= O\left(\int_{1/_{(n+m)}}^{1/_{(n+m)\delta}} \frac{|dh(t)|}{t} \right) \quad , \qquad \text{by lemma 3}$$

(3.5.9) $$= o(1) \text{ , as } n \to \infty ,$$

uniformly with respect to m.

Lastly, we have

$$Q_2 = \int_{1/_{(n+m)^\delta}}^{\pi} dh(t) \frac{1}{2\pi P_n} \sum_{v=0}^{n} p_{n-v} \frac{\sin(v+m+1)t - \sin mt}{2(v+1) \quad \sin^2(t/2)}$$

$$= \int_{1/_{(n+m)^\delta}}^{\pi} \frac{1}{2\pi P_n} \sum_{v=0}^{n} p_{n-v} \frac{\sin(v+m+1)t}{2(v+1) \quad \sin^2(t/2)} dh(t)$$

$$- \int_{1/_{(n+m)^\delta}}^{\pi} \frac{1}{2\pi P_n} \sum_{v=0}^{n} p_{n-v} \frac{\sin\ mt}{2(v+1) \quad \sin^2(t/2)} dh(t)$$

(3.5.10) $= Q_{21} - Q_{21}$, say.

Now, using second mean value theorem, we have

$$Q_{2.1} \leq \frac{1}{2\pi P_n} \sum_{\nu=0}^{n} p_{n-\nu} \frac{1}{2(\nu+1)} \sin^2\left(\frac{1}{(n+m)^\delta}\right) \int_{1/(n+m)^\delta}^{\epsilon^1} \sin(\nu+m+1)t \, dh(t)$$

where $\frac{1}{(n+m)^\delta} < \epsilon^1 < \pi$

(3.5.11) $Q_{2.1} = o(1)$, as $n \to \infty$,

uniformly with respect to m.

Similarly

$$Q_{2.1} = \frac{1}{2\pi P_n} \int_{1/(n+m)^\delta}^{\pi} \sum_{\nu=0}^{n} p_{n-\nu} \frac{\sin mt}{2(\nu+1)\sin^2(t/2)} dh(t)$$

$$\leq \frac{1}{4\pi P_n} \sum_{\nu=0}^{n} \frac{p_{n-\nu}}{(\nu+1)} \frac{1}{\sin^2\left(\frac{1}{(n+m)^\delta}\right)} \int_{1/(n+m)^\delta}^{\epsilon^1} \sin mt \, dh(t)$$

where $\frac{1}{(n+m)^\delta} < \epsilon^1 < \pi$

(3.5.12) $Q_{2.2} = o(1)$, as $n \to \infty$,

uniformly with respect to m.

Now from (3.5.11) and (3.5.12), we have

(3.5.13) $Q_2 = o(1)$, as $n \to \infty$, uniformly w.r.t.m.

From (3.5.9) and (3.5.13), we get

$$\int_{1/(n+m)^\delta}^{\pi} \overline{N}(t) \, dh(t) = o(1) \text{ , as } n \to \infty \text{ ,}$$

uniformly with respect to m.

This completes the proof of the theorem 2.

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