CHAPTER – 2

ON ALMOST $(\overline{N}, p_n)$ SUMMABILITY OF FOURIER SERIES AND ITS CONJUGATE SERIES

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2.1 DEFINITIONS AND NOTATIONS:

Let $\{s_n\}$ be the sequence of partial sums of a given infinite series $\sum a_n$. A bounded sequence $\{s_n\}$ is said to be almost convergent to a finite limit s, if

(2.1.1) $$\lim_{n\to\infty} \frac{1}{n+1}\sum_{k=m}^{n+m} s_k = s \ ,$$

uniformly with respect to m.

Let $\{p_n\}$ be a sequence of non-zero real constants with R_n as its non-vanishing n-th partial sum.

We define for the first time that the series $\sum a_n$, or the sequence $\{s_n\}$ is said to be almost $(\overline{N},p_n)$ summable to s, if

(2.1.2) $$\lim_{n\to\infty} t_{n,m} = \frac{1}{R_n}\sum_{k=0}^{n} p_k s_{k,m} = s \ ,$$

uniformly with respect to m, where

(2.1.3) $$s_{k,m} = \frac{1}{k+1}\sum_{v=m}^{k+m} s_v \ .$$

Let f (t) be a 2π-periodic and Lebesgue integrable function of t in the interval $(-\pi,\pi)$. Then the Fourier series of f(t) is given by

(2.1.4) $$f(t) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt) = \sum_{n=0}^{\infty} A_n(t) \ .$$

and its conjugate series is given by

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(2.1.5) $$\sum_{n=1}^{\infty}(b_n \cos nt - a_n \sin nt) = \sum_{n=1}^{\infty} B_n(t) \ .$$

Let us write, with a fixed point x,

$$\phi(t) = f(x+t) + f(x-t) - 2\, f(x),$$

$$\psi(t) = f(x+t) - f(x-t) \ ,$$

$$N_{n,m}(t) = \frac{1}{2\pi R_n} \sum_{k=0}^{n} p_k \frac{\cos mt - \cos(k+m+1)t}{2(k+1) \ \sin^2 \frac{t}{2}} \ ,$$

$$\bar{N}_{n,m}(t) = \frac{1}{2\pi R_n} \sum_{k=0}^{n} p_k \frac{\sin mt - \sin(k+m+1)t}{2(k+1) \ \sin^2 \frac{t}{2}} \ ,$$

$$\Phi(t) = \int_0^t |\phi(u)| \, du \ ,$$

$$\Psi(t) = \int_0^t |\psi(u)| \, du \ ,$$

and $$\tau = \left[\frac{1}{t}\right] = \text{the integral part of } \frac{1}{t} .$$

2.2 The object of the present chapter is to establish the following two theorems on almost $(\bar{N}, p_n)$ summability of Fourier series (2.1.4) and its conjugate series (2.1.5).

THEOREM 1. Let $\{p_n\}$ be a non-negative, monotonic non-increasing sequence of real constants such that its n-th partial sum $R_n \to \infty$, as $n \to \infty$.

Let $\lambda(t)$ and $K(t)$ be two positive functions of t such that $\lambda(t)$, $K(t)$ and $\frac{t.\lambda(t)}{K(t)}$ increase monotonically with t and

(2.2.1) $$\lambda(n+m) \ R_{n+m} = O\left[\{K(R_{n+m})\}^{\delta}\right], \text{ as } n \to \infty,$$

uniformly with respect to m.

If

$$(2.2.2) \qquad \Phi(t) = \int_0^t |\phi(u)| \; du = o\left[\frac{\lambda\left(\frac{1}{t}\right)p_\tau}{\{K(R_\tau)\}^\delta}\right], \quad \text{as} \quad t \to +0,$$

and

$$(2.2.3) \qquad \int_{1/(n+m)}^{1/(n+m)^\delta} \frac{|\phi(u)|}{u} \; du = o(1), \quad \text{as} \quad n \to \infty,$$

where $0 < \delta < 1$, uniformly with respect to m, then the series (2.1.4) is almost $(\overline{N}, p_n)$ summable to f(x) at the point t = x.

THEOREM 2. Let the sequence $\{p_n\}$ and the functions $\lambda(t)$ and K (t) be the same as in theorem 1. If

$$(2.2.4) \qquad \Psi(t) = \int_0^t |\psi(u)| \; du = o\left[\frac{\lambda\left(\frac{1}{t}\right)p_\tau}{\{K(R_\tau)\}^\delta}\right], \quad \text{as} \quad t \to +0,$$

and

$$(2.2.5) \qquad \int_{1/(n-m)}^{1/(n-m)^\delta} \frac{|\psi(u)|}{u} \; du = o(1), \quad \text{as} \quad n \to \infty,$$

where $0 < \delta < 1$, uniformly with respect to m, then the conjugate series (2.1.5) is almost $(\overline{N}, p_n)$ summable to

$$(2.2.6) \qquad \overline{f}(x) = \frac{1}{2\pi} \int_0^\pi \psi(t) \; \cot\left(\frac{t}{2}\right) dt,$$

at every point t = x, where this integral exists.

2.3 For the proof of our theorems, we shall use the following lemmas :

LEMMA 1. Let us write

$$N_{n,m}(t)=\frac{1}{2\pi R_n}\sum_{k=0}^{n} p_k \frac{\cos mt-\cos(k+m+1)t}{2(k+1)\ \sin^2\frac{t}{2}} \quad ,$$

then

$$N_{n,m}(t)=\begin{cases} O(n+m) \ , & \text{for } 0<t\le\frac{1}{(n+m)} \\ O\left(\frac{1}{t}\right) \ , & \text{for } \frac{1}{(n+m)}<t\le\pi \ . \end{cases}$$

PROOF OF LEMMA 1. We have

$$N_{n,m}(t)=\frac{1}{2\pi R_n}\sum_{k=0}^{n} p_k \frac{\sin\left(m+\frac{k+1}{2}\right)t\,.\sin\left(\frac{k+1}{2}\right)t}{(k+1)\ \sin^2\frac{t}{2}}$$

$$=O\left(\frac{1}{R_n}\right)\sum_{k=0}^{n} p_k(2m+k+1)$$

$$=O(n+m)\ ,\ \text{for}\ \ 0<t\le\frac{1}{(n+m)} \ .$$

Similarly on expending sine and cosine in powers of t, we get

$$N_{n,m}(t)=O\left(\frac{1}{t}\right),\ \text{for}\ \frac{1}{n+m}<t\le\pi.$$

LEMMA 2. Let

$$\overline{N}_{n,m}(t)=\frac{1}{2\pi R_n}\sum_{k=0}^{n} p_k \frac{\sin\ mt-\sin(k+m+1)t}{2(k+1)\ \sin^2\frac{t}{2}}$$

then

$$N_{n,m}(t)=\begin{cases} O(n+m) \ , & \text{for } 0<t\le\frac{1}{(n+m)} \\ O\left(\frac{1}{t}\right) \ , & \text{for } \frac{1}{(n+m)}<t\le\pi. \end{cases}$$

PROOF OF LEMMA 2. The proof will follow in the proof of lemma 1.

2.4 PROOF OF THEOREM 1.

The n-th partial sum of the series (2.1.4) is given by

$$s_n(x) - f(x) = \frac{1}{2\pi}\int_0^\pi \phi(t)\frac{\sin\left(m+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\,dt,$$

so that

$$s_{k,n} - f(x) = \frac{1}{(k+1)}\sum_{n-m}^{k+m}\{s_n - f(x)\}$$

$$= \frac{1}{2\pi(k+1)}\int_0^\pi \phi(t)\frac{\cos mt - \cos(k+m+1)t}{2\sin^2\frac{t}{2}}\,dt\,.$$

Therefore, by (2.1.2), we have

$$t_{n,m} - f(x) = \frac{1}{R_n}\sum_{k-0}^{n} p_k\{s_{k,m} - f(x)\}$$

$$= \frac{1}{2\pi R_n}\int_0^\pi \phi(t)\sum_{k=0}^{n} p_k\frac{\cos mt - \cos(k+m+1)t}{2\,(k+1)\sin^2\frac{t}{2}}\,dt$$

$$= \int_0^\pi \phi(t)N_{n,m}(t)\,dt$$

$$= \left[\int_0^{1/(n+m)} + \int_{1/(n+m)}^{1/(n+m)^\delta} + \int_{1/(n+m)^\delta}^{\pi}\right]\phi(t)\,N_{n,m}(t)dt$$

(2.4.1) $\qquad = I_1 + I_2 + I_3$, say.

In order to prove our theorem, we have to show that

(2.4.2) $\qquad t_{n,m} - f(x) = \int_0^\pi \phi(t)\,N_{n,m}(t)\,dt = o(1)$, as $n \to \infty$

uniformly with respect to m.

Let us first consider I_1. Now

$$|I_1| = \left[\int_0^{1/(n+m)} |\phi(t)||N_{n,m}(t)|\ dt\right]$$

$$= O(n+m)\ \int_0^{1/(n+m)} |\phi(t)|\ dt$$

$$= O(n+m)\ o\left[\frac{\lambda(n+m)\ p_{n+m}}{\{K(R_{n+m})\}^{\delta}}\right], \text{ by } (2.2.2)$$

$$= o\left[\frac{\lambda(n+m)\ R_{n+m}}{\{K(R_{n+m})\}^{\delta}}\right], \quad \text{since } n\,p_n \le R_n\ ,$$

(2.4.3) $\quad = o(1)$, as $n \to \infty$, by (2.2.1)

uniformly with respect to m.

Next, considering I_2, we have by (2.2.3) and lemma 1,

$$|I_2| = O\left[\int_{1/(n+m)}^{1/(n+m)^{\delta}} \frac{|\phi(t)|}{t}\ dt\right]$$

(2.4.4) $\quad = o(1)$, as $n \to \infty$,

uniformly with respect to m.

Lastly, we have

$$I_3 = \frac{1}{2\pi R_n}\int_{1/(n+m)^{\delta}}^{\pi} \phi(t)\sum_{k=0}^{n} p_k \frac{\cos mt - \cos\ (k+m+1)t}{2(k+1)\sin^2\frac{t}{2}}dt$$

$$= \frac{1}{2\pi R_n}\int_{1/(n+m)^{\delta}}^{\pi} \phi(t)\sum_{k=0}^{n} p_k \frac{\cos mt}{2(k+1)\sin^2\frac{t}{2}}dt$$

$$- \frac{1}{2\pi R_n}\int_{1/(n+m)^{\delta}}^{\pi} \phi(t)\sum_{k=0}^{n} p_k \frac{\cos\ (k+m+1)t}{2(k+1)\sin^2\frac{t}{2}}dt$$

$$= I_{31} - I_{32}\ , \text{ say.}$$

Now, using second mean value theorem,

$$I_{3,1} \leq \frac{1}{R_n}\sum_{k=0}^{n} p_k \frac{1}{2\sin^2\left\{\frac{1}{2(n+m)^\delta}\right\}} \int_{1/(n+m)^\delta}^{\epsilon} \phi(t)\ \cos mt\, dt\ ,$$

where $\frac{1}{(n+m)^\delta} < \epsilon < \pi$,

(2.4.5) $\quad = o\,(1)$, as $n \to \infty$,

uniformly with respect to m.

Similarly,

$$I_{3,2} = \frac{1}{2\pi\, R_n} \int_{1/(n+m)^\delta}^{\pi} \phi(t) \sum_{k=0}^{n} p_k \frac{\cos(k+m+1)t}{2(k+1)\sin^2\frac{t}{2}} dt$$

$$\leq \frac{1}{2\sin^2\left\{\frac{1}{2(n+m)^\delta}\right\}} \int_{1/(n+m)^\delta}^{\pi} |\phi(t)|\ dt$$

(2.4.6) $\quad = o\,(1)$, as $n \to \infty$,

uniformly with respect to m.

Collecting from (2.4.2) to (2.4.6), we get

$I_3 = o\,(1)$, as $n \to \infty$,

uniformly with respect to m.

This completes the proof of theorem 1.

2.5 PROOF OF THEOREM 2.

Let $\bar{s}_n$ be the n-th partial sum of the series (2.1.5). Then it is easy to show that

$$\bar{s}_n(x) - \bar{f}(x) = -\frac{1}{\pi}\int_0^{\pi} \psi(t) \frac{\cos\left(n+\frac{1}{2}\right)t}{2\sin^2\frac{t}{2}}\ dt\ ,$$

so that

$$\bar{s}_{k,m} - \bar{f}(x) = \frac{1}{k+1}\sum_{n=m}^{k+m}\left\{\bar{s}_n(x) - \bar{f}(x)\right\}$$

$$= -\frac{1}{\pi(k+1)}\int_0^\pi \psi(t)\sum_{n=m}^{k+m}\frac{\cos\left(n+\frac{1}{2}\right)t}{2\sin^2\frac{t}{2}}\,dt$$

$$= \frac{1}{2\pi(k+1)}\int_0^\pi \psi(t)\frac{\sin mt - \sin(k+m+1)t}{2\sin^2\frac{t}{2}}\,dt \, .$$

and , therefore, we have

$$\bar{t}_{n,m} - \bar{f}(x) = \frac{1}{2\pi R_n}\int_0^\pi \psi(t)\sum_{k=0}^{n} p_k \frac{\sin mt - \sin(k+m+1)t}{2(k+1)\sin^2\frac{t}{2}}\,dt$$

$$= \int_0^\pi \psi(t)\bar{N}_{n+m}(t)\,dt, \text{ say} .$$

Now, in order to prove the theorem, we are required to show that

$$\bar{t}_{n,m} - \bar{f}(x) = o(1) \, , \quad \text{as} \quad n \to \infty \, ,$$

uniformly with respect to m, at every point x, where the integral (2.2.6) exists.

The proof will completely follow as in theorem 1. It is interesting to note that our theorem improve the results of Ganguly (1979).

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