APPENDIX

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ON ALMOST ($\bar{N}$, p_n) SUMMABILITY OF FOURIER SERIES AND ITS CONJUGATE SERIES

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ıper, two theorems on almost $(\bar{N}, p_n)$ summability of Fourier series and its conjugate ve been established.

'INITIONS AND NOTATIONS : Let $\{s_n\}$ be the sequence of partial sums of a finite series $\sum a_n$. A bounded sequence $\{s_n\}$ is said to be almost convergent to a ıit s, if

$$\lim_{n\to\infty} \frac{1}{n+1}\sum_{k=m}^{n+m} s_k = s, \qquad \dots (1.1)$$

y with respect to m.

be a sequence of non-zero real constants with R_n as its non-vanishing n–th partial : define for the first time that the series $\sum a_n$, or the sequence $\{s_n\}$ is said to be $\bar{N}, p_n)$ summable to s, if

$$\lim_{n\to\infty} t_{n,m} = \frac{1}{R_n}\sum_{k=0}^{n} p_k s_{k,m} = s, \qquad \dots (1.2)$$

y with respect to m,

$$s_{k,m} = \frac{1}{k+1}\sum_{v=m}^{k+m} s_v. \qquad \dots (1.3)$$

ɛ a 2π-periodic and Lebesgue-integrable function of t in the interval $(-\pi, \pi)$.

Fourier series of $f(t)$ is given by

$$f(t) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$$

$$= \sum_{n=0}^{\infty} A_n(t) \qquad \dots (1.4)$$

ınjugate series is given by

$$\sum_{n=1}^{\infty} (b_n \cos nt - a_n \sin nt) = \sum_{n=1}^{\infty} B_n(t) \qquad \dots (1.5)$$

ite, with a fixed point x,

$$\phi(t) = f(x+t) + f(x-t) - 2f(x)$$

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$$\psi(t) = f(x+t) - f(x-t)$$

$$N_{n,m}(t) = \frac{1}{2\pi R_n} \sum_{k=0}^{n} p_k \frac{\cos mt - \cos (k+m+1)t}{2(k+1) \sin^2 t/2}$$

$$\bar{N}_{n,m}(t) = \frac{1}{2\pi R_n} \sum_{k=0}^{n} p_k \frac{\sin mt - \sin (k+m+1)t}{2(k+1) \sin^2 t/2}$$

$$\Phi(t) = \int_0^t |\phi(u)| du$$

$$\Psi(t) = \int_0^t |\psi(u)| du$$

and $\tau = \left[\frac{1}{t}\right]$ = the integral part of $\frac{1}{t}$.

2. The object of the present paper is to establish the following two theorems on alm $(\bar{N}, p_n)$ summability of Fourier series and its conjugate series :

Theorem 1 : Let $\{p_n\}$ be a non-negative, monotonic non-increasing sequence of r constants such that its n–th partial sum $R_n \to \infty$ as $n \to \infty$.

Let $\lambda(t)$ and $K(t)$ be two positive functions of t such that $\lambda(t)$, $K(t)$

and $$\frac{t.\lambda(t)}{K(t)}$$

increase monotonically with t and

$$\lambda(n+m)\, R_{n+m} = O\left[\{K(R_{n+m})\}^{\delta}\right], \text{ as } n \to \infty \quad \dots (2$$

uniformly with respect to m.

If $$\Phi(t) = \int_0^t |\phi(u)| du = o\left[\frac{\lambda\left(\frac{1}{t}\right) p_\tau}{\{K(R_\tau)\}^{\delta}}\right], \text{ as } t \to +0 \quad \dots (2$$

and $$\int_{\frac{1}{n+m}}^{\frac{1}{(n+m)^{\delta}}} \frac{|\phi(u)|}{u} du = o(1), \text{ as } n \to \infty \quad \dots (2$$

where $0 < \delta < 1$, uniformly with respect to m.

then the series (1.4) is almost $(\bar{N}, p_n)$ summable to $f(x)$ at the point $t = x$.

Theorem 2 : Let the sequence $\{p_n\}$ and the functions $\lambda(t)$ and $K(t)$ be the same as theorem 1.

$$\Psi(t)=\int_0^t |\psi(u)|\,du=o\left[\frac{\lambda\left(\frac{1}{t}\right)p_\tau}{\{K(R_\tau)\}^\delta}\right],\ \text{as } t\to+0 \qquad \dots (2.4)$$

$$\int_{\frac{1}{n+m}}^{\frac{1}{(n+m)^\delta}} \frac{|\psi(u)|}{u}\,du=o(1),\ \text{as } n\to\infty \qquad \dots (2.5)$$

$0<\delta<1$, uniformly with respect to m,

ɜ conjugate series (1.5) is almost $(\overline{N}, p_n)$ summable to

$$\overline{f}(x)=\frac{1}{2\pi}\int_0^\pi \psi(t)\ \cot\ \frac{t}{2}dt \qquad \dots (2.6)$$

/ point $t=x$, where this integral exists.

ƆR THE PROOF OF OUR THEOREMS, WE SHALL USE THE FOLLOWING [AS :

ı 1. Let us write

$$N_{n,m}(t)=\frac{1}{2\pi R_n}\sum_{k=0}^{n} p_k\frac{\cos mt-\cos(k+m+1)t}{2(k+1)\ \sin^2\frac{t}{2}}$$

$$N_{n,m}(t)=\begin{cases} O(n+m), & \text{for } 0<t\le\frac{1}{n+m} \\ O\left(\frac{1}{t}\right), & \text{for } \frac{1}{n+m}<t\le\pi \end{cases}$$

f Lemma 1. We have

$$N_{n,m}(t)=\frac{1}{2\pi R_n}\sum_{k=0}^{n} p_k\frac{\sin\left(m+\frac{k+1}{2}\right)t\ \sin\left(\frac{k+1}{2}\right)t}{(k+1)\ \sin^2 t/2}$$

$$=O\left(\frac{1}{R_n}\right)\sum_{k=0}^{n} p_k(2m+k+1)$$

$$=O(n+m),\ \text{for } o<t\le\frac{1}{(n+m)}$$

/ on expending sine and cosine in powers of t,

$$N_{n,m}(t)=O(1/t),\ \text{for } \frac{1}{n+m}<t\le\pi .$$

2. Let
$$\overline{N}_{n,m}(t)=\frac{1}{2\pi R_n}\sum_{k=0}^{n} p_k\frac{\sin\ mt-\sin\ (k+m+1)\ t}{2(k+1)\ \sin^2\ t/2}$$

then $$N_{n,m}(t)=\begin{cases} O\ (n+m), & \text{for } o<t\le \dfrac{1}{(n+m)} \\ O\ (1/t), & \text{for } \dfrac{1}{n+m}<t\le \pi \end{cases}$$

Proof of Lemma 2. The Proof will follow in the proof of lemma 1.

4. PROOF OF THEOREM 1. The n–th partial sum of the series (1.4) is given by

$$s_n(x)-f(x)=\frac{1}{2\pi}\int_o^{\pi}\phi(t)\ \frac{\sin\left(m+\frac{1}{2}\right)t}{\sin\ t/2}dt$$

so that $$s_{k,n}-f(x)=\frac{1}{k+1}\sum_{n=m}^{k+m}\{s_n-f(x)\}$$

$$=\frac{1}{2\pi(k+1)}\int_o^{\pi}\phi(t)\frac{\cos\ mt-\cos\ (k+m+1)t}{2\ \sin^2\ t/2}dt.$$

Therefore, by (1.2), we have

$$t_{n,m}-f(x)=\frac{1}{R_n}\sum_{k=0}^{n}p_k\ \{s_{k,m}-f(x)\}$$

$$=\frac{1}{2\pi R_n}\int_o^{\pi}\phi(t)\sum_{k=0}^{n}p_k\frac{\cos\ mt-\cos\ (k+m+1)t}{2(k+1)\ \sin^2\ t/2}dt.$$

$$=\int_o^{\pi}\phi(t)\ N_{n,m}(t)dt$$

$$=\left[\int_0^{1/(n+m)}+\int_{1/(n+m)}^{1/(n+m)^{\delta}}+\int_{1/(n+m)^{\delta}}^{\pi}\right]\phi(t)N_{n,m}(t)dt$$

$$=I_1+I_2+I_3,\ \text{say} \qquad \ldots$$

In order to prove our theorem, we have to show that

$$t_{n,m}-f(x)=\int_o^{\pi}\phi(t)N_{n,m}(t)dt=o(1),\ \text{as } n\to\infty \qquad \ldots$$

uniformly with respect to m.

Let us first consider I_1.

Now $$|I_1|=\left[\int_0^{1/(n+m)}|\phi(t)||N_{n,m}(t)|dt\right],$$

$$=O(n+m)\int_0^{1/(n+m)}|\phi(t)|dt$$

$$=O(n+m)\ o\left[\frac{\lambda(n+m)p_{n+m}}{\{K(R_{n+m})\}^{\delta}}\right]\ \text{by (2.2)}$$

$$= o\left[\frac{\lambda(n+m)R_{n+m}}{\{K(R_{n+m})\}^{\delta}}\right], \text{since } np_n \le R_n, \qquad \ldots (4.3)$$

$$= o\ (1), \text{ as } n \to \infty$$

formly with respect to m,

:t, considering I_2, we have by (2.3) and lemma 1.

$$|I_2| = O\left[\int_{1/(n+m)}^{1/(n+m)^{\delta}} \frac{|\phi(t)|}{t} dt\right] \qquad \ldots (4.4)$$

$$= o\ (1), \text{ as } n \to \infty$$

ormly with respect to m.

ly, we have

$$I_3 = \frac{1}{2\pi R_n}\int_{1/(n+m)^{\delta}}^{\pi} \phi(t)\sum_{k=0}^{n} p_k \frac{\cos mt - \cos(k+m+1)t}{2(k+1)\sin^2 t/2} dt$$

$$= \frac{1}{2\pi R_n}\int_{1/(n+m)^{\delta}}^{\pi} \phi(t)\sum_{k=0}^{n} p_k \frac{\cos mt}{2(k+1)\sin^2 t/2} dt$$

$$- \frac{1}{2\pi R_n}\int_{1/(n+m)^{\delta}}^{\pi} \phi(t)\sum_{k=0}^{n} p_k \frac{\cos(k+m+1)t}{2(k+1)\sin^2 t/2} dt$$

$$= I_{3.1} - I_{3.2}, \text{ say.}$$

, using second mean value theorem,

$$I_{3.1} \le \frac{1}{R_n}\sum_{k=0}^{n} p_k \frac{1}{2\sin^2\left\{\frac{1}{2(n+m)^{\delta}}\right\}}\int_{1/(n+m)^{\delta}}^{\varepsilon} \phi(t)\cos mt\, dt,$$

: $\frac{1}{(n+m)^{\delta}} < \varepsilon < \pi,$

$$= o\ (1), \text{ as } n \to \infty \qquad \ldots (4.5)$$

·mly with respect to m.

irly, $I_{3.2} = \frac{1}{2\pi R_n}\int_{1/(n+m)^{\delta}}^{\pi} \phi(t)\sum_{k=0}^{n} p_k \frac{\cos(k+m+1)t}{2(k+1)\sin^2 t/2} dt$

$$\le \frac{1}{2\sin^2\left\{\frac{1}{2(n+m)^{\delta}}\right\}}\int_{1/(n+m)^{\delta}}^{\pi} |\phi(t)| dt$$

$$= o\ (1), \text{ as } n \to \infty \qquad \ldots (4.6)$$

uniformly with respect to m.

Collecting from (4.2) to (4.6), we get

$$I_3 = o\ (1),\ \text{as } n \to \infty$$

uniformly with respect to m.

This completes the proof of theorem 1.

5. PROOF OF THEOREM 2. Let $\bar{s}_n$ be the n–th partial sum of the series (1.5). Then it easy to show that

$$\bar{s}_n(x) - \bar{f}(x) = -\frac{1}{\pi}\int_o^{\pi} \psi(t)\frac{\cos\left(n+\frac{1}{2}\right)t}{2\sin^2 t/2}dt,$$

so that

$$\bar{s}_{k,m} - \bar{f}(x) = \frac{1}{k+1}\sum_{n=m}^{k+m} \{\bar{s}_n(x) - \bar{f}(x)\}$$

$$= \frac{-1}{\pi(k+1)}\int_0^{\pi} \psi(t)\sum_{n=m}^{k+m} \frac{\cos\left(n+\frac{1}{2}\right)t}{2\sin^2 t/2}dt$$

$$= \frac{1}{2\pi(k+1)}\int_0^{\pi} \psi(t)\frac{\sin mt - \sin(k+m+1)t}{2\sin^2 t/2}dt$$

and, therefore, we have

$$\bar{t}_{n,m} - \bar{f}(x) = \frac{1}{2\pi R_n}\int_0^{\pi} \psi(t)\sum_{k=0}^{n} p_k \frac{\sin mt - \sin(k+m+1)t}{2(k+1)\sin^2 t/2}dt$$

$$= \int_0^{\pi} \psi(t)\,\bar{N}_{n,m}(t)\,dt, \text{ say.}$$

Now, in order to prove the theorem, we are required to show that

$$\bar{t}_{n,m} - \bar{f}(x) = o\ (1),\ \text{as } n \to \infty$$

uniformly with respect to m, at every point x, where the integral (2.6) exists.

The proof will completely follow as in theorem 1.

It is interesting to note that our theorem improve the result of Ganguly (1979).

REFERENCES :

1. Chaubey, V.S., *Almost summability of conjugate series of Fourier series.* Thesis, V.B.S. Purvan University, Jaunpur (2002).
2. Ganguly, A., On (N, p_n) summability of Fourier series and its conjugate series, *Indian Jour. Pure A Math.*, (10), **10**, 1228-1235 (1979).
3. Singh, T., On summability of Fourier series and its conjugate series, proceedings of the National Inst of science of India, (4), No. **1**, 29, 65-73 (1963).

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NIFORM TRIANGULAR MATRIX SUMMABILITY OF LEGENDRE SERIES

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ıer, a theorem on uniform triangular matrix summability of Legendre series has !ished.

:TIONS AND NOTATIONS : Let $\sum_{n=0}^{\infty} u_n(x)$ be an infinite series with $\{s_n(x)\}$ as :e of its n–th partial sums. Let $(\lambda_{n,k})$ $(n = 0, 1, 2 \ldots\ldots, k = 0, 1, 2 \ldots\ldots n;\ \lambda_{n,0} = 1)$ .ular martix of real or complex numbers.

$$\sigma_n(x) = \sum_{k=0}^{n} \lambda_{n,k} u_k(x) = \sum_{k=0}^{n} \Delta\lambda_{n,k} s_k(x) \qquad \ldots (1.1)$$

$$\Delta\lambda_{n,k} = \lambda_{n,k} - \lambda_{n,k+1}$$

$$\Delta^2\lambda_{n,k} = \Delta\lambda_{n,k} - \Delta\lambda_{n,k+1}$$

$$\sigma_n(x) \to s(x), \text{ as } n \to \infty \qquad \ldots (1.2)$$

y that the series $\sum u_n(x)$ is summable (^) to $s(x)$ at a point x.

$$\sigma_n(x) - s(x) = o(1), \text{ as } n \to \infty. \qquad \ldots (1.3)$$

in a set E,

y that the series $\sum u_n(x)$ is summable (^) uniformly in set E to the sum $s(x)$.

ır, if

$$\Delta\lambda_{n,k} = \begin{cases} \left[(n+1-k)\log n\right]^{-1} & ,k \le n \\ 0 & ,k > n, \end{cases} \qquad \ldots (1.4)$$

:) defined by (1, 1) is same as the harmonic mean $\left(N, \frac{1}{n+1}\right)$ of the sequence

;endre series associated with a Lebesgue integrable function $f(x)$ in the range (– 1, 1)

$$f(x) \sim \sum_{n=0}^{\infty} a_n P_n(x) \qquad \ldots (2.1)$$

$$a_n = \left(n + \frac{1}{2}\right) \int_{-1}^{1} f(x)\, P_n(x) dx \qquad \ldots (2.2)$$

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and the n-th Legendre polynomial $P_n(x)$ is defined by the generating function

$$\frac{1}{\sqrt{1-2xz+z^2}} = \sum_{n=0}^{\infty} P_n(x) z^n \quad \ldots$$

We use the following notations :

$$\psi(t) = \psi_0(t) = f\{\cos(\theta - t) - f(\cos\theta)\} \quad \ldots$$

and
$$N_n(t) = \sum_{k=0}^{n} \Delta\lambda_{n,k} \frac{\sin\,(k+1)t}{\sin\,t/2} \quad \ldots$$

3. Dwivedi (1970) has established the following theorem on uniform summability of Leg series,

Theorem A. If

$$\int_0^t |f(x \pm u) - f(x)|\,du = o\left[\frac{t}{\log\left(\frac{1}{t}\right)}\right], \text{ as } t \to +0$$

uniformly in a set E defined in interval (–1, 1) in which $f(x)$ is bounded, then the series (2 summable by hormonic means uniformly in E to the sum $f(x)$.

In the present paper, we propose to extend the above result for uniform triangular m summability of Legendre series by proving the following ;

Theorem. If

$$\int_t^{\eta} \frac{|\psi(u)|}{u^2}\,du = o\left[\frac{\lambda\left(\frac{1}{t}\right)}{tP\left(\frac{1}{t}\right)}\right], \text{ as } t \to +0 \quad \ldots$$

where $0 \le t \le \eta < \pi$ is fixed uniformly in a set E defined in the interval (– 1, 1) in whic is bounded and is a positive non-decreasing function of t, such that and

$$\lambda(n)\log n = O(P_n), \text{ as } n \to \infty \quad \ldots$$

then the series (2.1) is summable (^) uniformly in a set E to the sum $f(x)$.

Note : It is worth noticing that our condition (3.2) is less stronger than the condition (3 theorem A in the following sense.

Following on the lines Foa (1943), it may be easily proved that under condition (3.1), we

$$\int_0^t \left|f\{\cos(\theta - y)\} - f(\cos\theta)\right| dy = o\left[\frac{t\lambda\left(\frac{1}{t}\right)}{P_{(1/t)}}\right], \text{ as } t \to 0 \quad \ldots$$

where $x = \cos\theta,\ x + u = \cos\phi,\ \theta - \phi = y;$

so that in view of (2.4)

$$\int_0^t |\psi(y)|\,dy = o\left[\frac{t\lambda\left(\frac{1}{t}\right)}{P_{(1/t)}}\right], \text{ as } t \to 0 \qquad \ldots (3.5)$$

ı be easily seen, as under that (3.2) implies (3.5) on integrating by parts, we have

$$\int_0^t |\psi(y)|\,dy = \int_0^t y^2 \frac{|\psi(y)|}{y^2}\,dy$$

$$= y^2 \int_0^t \frac{|\psi(y)|}{y^2}\,dy - 2\int_0^t y\left[\frac{|\psi(y)|}{y^2}\,dy\right]dy$$

$$= \left[y^2 o\left(\frac{\lambda\left(\frac{1}{y}\right)}{yP_{(1/y)}}\right)\right]_0^t - 2\int_0^t y\, o\left(\frac{\lambda\left(\frac{1}{y}\right)}{yP_{(1/y)}}\right)dy$$

$$= o\left(\frac{y\lambda\left(\frac{1}{y}\right)}{P_{(1/y)}}\right)_0^t - 2\int_0^t o\left(\frac{\lambda\left(\frac{1}{y}\right)}{P_{(1/y)}}\right)dy$$

$$= o\left(\frac{t\,\lambda\left(\frac{1}{y}\right)}{P_{(1/y)}}\right), \text{ as } t \to +0$$

ourse of the proof of our theorem, we shall use the following lemmas :

$$\sum_{v=0}^{n}(2v+1)\,P_v(x)\,P_v(y) = \frac{(n+1)\,P_{n+1}(y)P_n(x) - P_{n+1}(x)P_n(y)}{(y-x)} \qquad \ldots (4.1)$$

y is known as Christoffel's formula of summation.

!. If $\{\Delta\lambda_{n,k}\}_{k=0}^{n}$ is a non-negative and non-decreasing sequence with respect to k, $\leq a \leq b \leq \infty$, $0 < t \leq \pi$ for any n.

$$\left|\sum_{k=a}^{b}\Delta\lambda_{n,n-k}e^{i(n-k)t}\right| = O\left[\frac{1}{t}\Delta_{n,n-\tau}\right], \qquad \ldots (4.2)$$

he integral part of $\frac{1}{t}$.

LEMMA 3. If $\{\Delta\lambda_{n,k}\}_{k=0}^{n}$ is a non-negative and non-decreasing sequence with respect such that

$$\sum_{k=0}^{n}\Delta\lambda_{n,k}=1,$$

then, as $n\to\infty$

$$\Delta\lambda_{n,k}=O\left(\frac{1}{n-k+1}\right)$$

uniformly for all $k\le n$, so that we get

$$\Delta\lambda_{n,0}=O\left(\frac{1}{n}\right) \qquad \ldots$$

LEMMA 4. If

$$N_n(t)=\sum_{k=0}^{n}\Delta\lambda_{n,k}\frac{\sin(k+1)t}{\sin\left(\frac{t}{2}\right)},$$

then $|N_n(t)|=O(n)$, as $n\to\infty$

uniformly in $0<t\le\frac{1}{n}$.

PROOF OF LEMMA 4. We have

$$|N_n(t)|=\left|\sum_{k=0}^{n}\Delta\lambda_{n,k}\frac{\sin(k+1)t}{\sin\left(\frac{t}{2}\right)}\right|$$

$$=O\left[\sum_{k=0}^{n}|\Delta\lambda_{n,k}|(k+1)\right]$$

Now, using Abel's transformation, we get

$$|N_n(t)|=O\left[\sum_{k=0}^{n-1}|k+1-k-2|\sum_{v=0}^{k}|\Delta\lambda_{n,v}|+(n+1)\sum_{k=0}^{n}|\Delta\lambda_{n,k}|\right]$$

$$=O\left[\sum_{k=0}^{n-1}\left\{\sum_{v=0}^{k}|\Delta\lambda_{n,v}|\right\}+(n+1)\sum_{k=0}^{n}|\Delta\lambda_{n,k}|\right]$$

By the regularity condition of (^) summability, there exists a constant M, such that

$$\sum_{k=0}^{\infty}|\Delta\lambda_{n,k}|<M, \text{ for any } n.$$

Therefore, we have

$$|N_n(t)|=O\left[Mn+(n+1)M\right]$$

$$= O(n), \text{ as } n \to \infty \qquad \dots (4.4)$$

y in $0 < t \le \frac{1}{n}$.

)F OF THE THEOREM : The n–th partial sum of the series (2.1) is

$$s_n(x) = \sum_{v=0}^{n} a_v P_v(x)$$

$$= \sum_{v=0}^{n} \frac{(2v+1)}{2} \int_{-1}^{1} f(y)\, P_v(y) P_v(x)\, dy, \text{ by (2.2)}$$

$$= \frac{(n+1)}{2} \int_{-1}^{1} \frac{P_{n+1}(y)\, P_n(x) - P_{n+1}(x) P_n(y)}{y-x} f(y), \text{ by (4.1)}$$

$(y) = 1$ it can be easily seen that

$$1 = \frac{(n+1)}{2} \int_{-1}^{1} \frac{P_{n+1}(y)\, P_n(x) - P_{n+1}(x) P_n(y)}{y-x} d(y)$$

ɜ,

$$s_n(x) - f(x) = \frac{(n+1)}{2} \int_{-1}^{1} [f(y) - f(x)] . \frac{P_{n+1}(y)\, P_n(x) - P_{n+1}(x) P_n(y)}{y-x} dy$$

ıke a positive number $s < 1$ and consider it as the sum of two other positive number α ‿et δ be another positive number such that $0 < \delta < \alpha$. αx and $\alpha x'$ be two continuous ıs of x with in $(-1, 1)$ which lie within the limits $\delta \le \alpha x \le \alpha, \delta \le \alpha x' \le \alpha$.

re,

$$f(x) = \frac{(n+1)}{2} \left[\int_{-1}^{x-\alpha x} + \int_{x-\alpha x}^{x+\alpha x'} + \int_{x+\alpha x'}^{1} \right] [f(y) - f(x)] \frac{P_{n+1}(y)\, P_n(x) - P_n(y) P_{n+1}(x)}{y-x} dy$$

$$= A_n(x) + B_n(x) + C_n(x), \text{ say} \qquad \dots (5.1)$$

(1909) has shown that uniformly for $-1 + s \le x \le 1 - s$

$$\begin{cases} \lim_{n\to\infty} A_n(x) = 0 \\ \lim_{n\to\infty} C_n(x) = 0 \end{cases} \qquad \dots (5.2)$$

ɜ suppose that

$$x = \cos\theta,\ y = \cos\phi,\ 0 < \theta < \pi,\ 0 < \phi < \pi$$

$$1 - \beta = \cos p,\ 1 - (\alpha + \beta) = 1 - s = \cos\ (p + \sigma)$$

$$1 < \rho < \frac{\pi}{2},\ 0 < \sigma,\ \rho + \sigma < \frac{\pi}{2}$$

f n denotes the minimum of $[arc\ \cos\ u - arc\ \cos\ (u + \alpha)]$.

ı $(-1,\ 1-\alpha)$, we have on the lines of Sansone (1959).

$$B_n \cos\theta = \frac{(n+1)}{2}\int_{\theta-\eta}^{\theta+\eta}\left[f(\cos\phi)-f(\cos\theta)\right]\frac{P_{n+1}(\cos\phi)P_n(\cos\theta)-P_{n+1}(\cos\theta)\,P_n(\cos\phi)}{\cos\phi-\cos\theta}\text{si}$$

in which $(\rho+\sigma)\le\theta\le\pi-(\rho+\sigma),\ 0<\eta\le\sigma.$

With successive transformation, we get

$$B_n(\cos\theta)=D_n(\theta)+E_n(\theta) \qquad \ldots$$

where $$D_n(\theta)=\frac{1}{2\pi\sqrt{\sin\theta}}\int_{\theta-\eta}^{\theta+\eta}\frac{\left[f(\cos\phi)-f(\cos\theta)\right]}{\sin\frac{1}{2}(\theta-\phi)}\sin\{(n+1)(\theta-\phi)\}\sqrt{\sin\phi}\,d\phi$$

and obviously on the line of Sansone (1959)

$$E_n(\theta)=o(1),\ \text{as } n\to\infty$$

uniformly where x lies within $(-1+s,\ 1-s)$ *i.e.*, in the set E.

Putting $\theta-\phi=t$, we get

$$D_n(\theta)=\frac{1}{\pi\sqrt{\sin\theta}}\int_0^{\eta}\left[f\{\cos(\theta-t)\}-f(\cos\theta)\right].\frac{\sin(n+1)t}{\sin\left(\frac{t}{2}\right)}\sqrt{\sin(\theta-t)}\,dt \quad \ldots$$

So, we get from (5.1) to (5.4),

$$s_n(x)-f(x)=\frac{1}{\pi\sqrt{\sin\theta}}\int_0^{\eta}\left[f\{\cos(\theta-t)\}-f(\cos\theta)\right].\frac{\sin(n+1)t}{\sin\left(\frac{t}{2}\right)}\sqrt{\sin(\theta-t)}\,dt+o(1$$

$$=O\left[\int_0^{\eta}\left[f\{\cos(\theta-t)\}-f(\cos\theta)\right].\frac{\sin(n+1)t}{\sin\left(\frac{t}{2}\right)}dt\right]$$

$$=O\left[\int_0^{\eta}\psi(t).\frac{\sin(n+1)t}{\sin\left(\frac{t}{2}\right)}dt\right]+o(1) \qquad \ldots ($$

uniformly in E.

Now, if $\sigma_n(x)$ be the (^)-mean of the sequence $\{s_n(x)\}$ of partial sums of the series (2.1) by the application of (1.1), we have

$$\sigma_n(x)-f(x)=\sum_{k=0}^{n}\Delta\lambda_{n,k}\left[s_k(x)-f(x)\right]$$

$$= O\left[\int_0^{\eta}\left\{\sum_{k=0}^{n}\Delta\lambda_{n,k}\frac{\sin(k+1)t}{\sin\left(\frac{t}{2}\right)}\right\}\psi(t)dt\right]+o(1)$$

$$= O\left[\int_0^{\eta}|\psi(t)|\ |N_n(t)|dt\right]+o(1)$$

$$= O(n)\left[\left\{\int_0^{\frac{1}{n}}+\int_{\frac{1}{n}}^{\eta}\right\}|\psi(t)|\ |N_n(t)|dt\right]+o(1)$$

$$= O(I_1)+O(I_2)+o(1) \qquad \ldots (5.6)$$

y in E.

theorem will be established, if we show that

$$I_1 = o(1) \qquad \ldots (5.7)$$

$$I_2 = o(1), \text{ as } n \to \infty$$

y in E.

st consider I_1, we have by (4.4),

$$I_1 = O(n)\left[\int_0^{\frac{1}{n}}|\psi(t)|dt\right]$$

$$= O(n)\left[o\left(\frac{\lambda(n)}{n\,P_n}\right)\right]$$

$$= O(n)\left(\frac{\lambda(n)}{P_n}\right)$$

$$= o(1), \text{ as } n \to \infty \qquad \ldots (5.8)$$

nsidering I_2, we have

$$I_2 = \left[\int_{\frac{1}{n}}^{\eta}|\psi(t)|\ |N_n(t)|dt\right]$$

$$= \int_{\frac{1}{n}}^{\eta}|\psi(t)|\left|\sum_{k=0}^{n}\Delta_{n,k}\frac{\sin(k+1)t}{\sin\left(\frac{t}{2}\right)}\right|dt$$

$$= \int_{\frac{1}{n}}^{\eta}|\psi(t)|\left|\sum_{k=0}^{n}\Delta'\lambda_{n,n-k}\frac{\sin(n-k+1)t}{\sin\left(\frac{t}{2}\right)}\right|dt$$

$$\leq \int_{\frac{1}{n}}^{\eta} \frac{|\psi(t)|}{t} \left| I_m \sum_{k=0}^{n} \Delta \lambda_{n,n-k} e^{i(n-k+1)t} \right| dt$$

$$= O\left[\int_{\frac{1}{n}}^{\eta} \frac{|\psi(t)|}{t^2} \sum_{k=0}^{n} \Delta \lambda_{n,n-\tau} dt \right], \text{ by (4.2)}$$

$$= O\left[\int_{\frac{1}{n}}^{\eta} \frac{|\psi(t)|}{t^2} \Delta \lambda_{n,0} dt \right]$$

$$= O\left(\frac{1}{n}\right) \int_{\frac{1}{n}}^{\eta} \frac{|\psi(t)|}{t^2} dt, \text{ by (4.3)}$$

$$= O\left(\frac{1}{n}\right) o\left(\frac{n\lambda(n)}{P_n}\right)$$

$$= o\left(\frac{\lambda(n)}{P_n}\right)$$

$$= o(1), \text{ as } n \to \infty \qquad \ldots$$

uniformly in E.

Now combining (5.6), (5.7), (5.8) and (5.9) we get the required result.

This completes the proof of the theorem.

REFERNECES :

1. Dwivedi, G.K., On the uniform harmonic and Nörlund summability of Legendre series, *Ph.D.* Chapter IX, B.H.U., Varanasi (1970).
2. Foà, A., Sulla Sommabilita Forte della serie Di Legendre, *Bull. Uni. Math., Ital.*, (2) 5, 18-27 (1943
3. Hobson, E.W., On the representation by a series of Legendre's functions, proceeding of the l Mathematical Society, (2) 2, 24-29 (1909).
4. Sansone, G., *Orthogonal functions*, English Edition (1959).

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