CHAPTER – 10

ON THE DEGREE OF APPROXIMATION OF FUNCTIONS BELONGING TO THE CLASS Lip (α, p)

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10.1 DEFINITIONS AND NOTATIONS :

Let f be a 2π periodic function integrable $L_p(p > 1)$ and let

$$f \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

(10.1.1) $$= \sum_{n=0}^{\infty} A_n(x)$$

be its Fourier series.

A functon $f \in \text{Lip } \alpha$ if

(10.1.2) $$f(x+h) - f(x) = O(|h|^{\alpha}) \quad \text{for } 0 < \alpha \leq 1 ,$$

We define the norm $\| \cdot \|_p$ by

(10.1.3) $$\|f\|_p = \left\{ \int_0^{2\pi} |f(x)|^p dx \right\}^{1/p} , p \geq 1$$

and the degree of approximation $E_n(f)$ by

(10.1.4) $$E_n(f) = \min_{T_n} \|f - T_n\|_p$$

where $T_n(x)$ is a trigonometrical polynomial of degree n.

We say that $f \in \text{Lip}(\alpha, q)$ for $a \leq x \leq b$ if

(10.1.5) $$\left\{ \int_a^b |f(x+h) - f(x)|^q dx \right\}^{1/q} \leq A|h|^{\alpha} , 0 < \alpha \leq 1, \ q \geq 1$$

where A is some constant. (see Def. 5.38 of McFadden (1942)).

This chapter in the form of research paper has been accepted for publication in Acta Ciencia Indica.

We write

$$\phi(t) = f(x+t) - f(x-t) - 2f(x) .$$

Lorentz (1948) has defined:

DEFINITION L_1. A sequence $\{s_n\}$ is said to almost convergent to a limit s , if

$$\lim_{n\to\infty} \frac{1}{(n+1)} \sum_{k=p}^{n+p} s_k = s \quad , \tag{10.1.6}$$

uniformly with respect to p.

Recently Sharma et al. (1977) have defined almost Borel summability.

Sharma and Qureshi (1980) defined :

DEFINITION L_2. A series $\sum u_n$ with the sequence of partial sums $\{s_n\}$ is said to be almost Riesz summable to s, provided .

$$t_{n,p} = \frac{1}{P_n} \sum_{k=0}^{n} p_k s_{k,p} \to s \, , \quad \text{as} \quad n \to \infty \tag{10.1.7}$$

uniformly with respect to p,

where

$$s_{k,p} = \frac{1}{k+1} \sum_{\mu=p}^{k+p} s_\mu$$

and $\{p_n\}$ be a sequence of non-negative constants, such that $p_0 > 0$ and

$$P_n = p_0 + p_1 + p_2 + \ldots\ldots\ldots\ldots\ldots + p_n .$$

The Riesz means are regular if and only if $P_n \to \infty$ with n (see theorem 1.4.4 of Petersen (1966))

10.2 Let $\{p_n\}$ be a non-negative, non-increasing generating sequence for the (N, p_n) method such that

(10.2.1) $P_n \equiv P(n) = p_0 + p_1 + p_2 + \dots\dots\dots\dots + p_n \to \infty$, as $n \to \infty$.

We write

$$p(y) = p_{[y]} \text{ and } P(y) = P_{[y]}$$

where [y] as usual denotes the greatest integer less than y.

Sahney and Rao (1972 a , b) have proved the following theorem.

THEOREM A. If f is periodic and belongs to the class Lip (α,p) , $0 < \alpha \leq 1$ and let $\{p_n\}$ be defined as in (10.2.1) and

$$\left(\int_1^n \frac{(P(y))^q}{y^{(q\alpha+2-q)}} dy\right)^{1/q} = O\left(\frac{P(n)}{n^{(q\alpha+1/q-1)/q}}\right)$$

then $$\|f - t_n\|_p = O\left(\frac{1}{n^{(\alpha p - 1/p)}}\right)$$

where $$t_n = \frac{1}{P_n}\sum_{k=0}^{n} p_{n-k} s_k \ .$$

i.e. the (N, p_n) mean of the Fourier series (10.1.1).

We have prove the following theorem.

THEOREM . The degree of approximation of a periodic function f belonging to the class Lip (α,p) , for $0 < \alpha \leq 1$ by almost Riesz mean is given by

$$\|f - t_{n,p}\|_p = O\left\{\frac{1}{(P_n)^{\alpha - 1/p}}\right\}$$

where Riesz means are regular and $\frac{1}{p} + \frac{1}{q} = 1$ such that $1 \leq p \leq \infty$.

To prove the theorem we shall need the following lemma.

LEMMA . (McFadden (1942), lemma 5.40) If f belongs to Lip (α,q) on $[0,\pi]$. then $\phi(t)$ also belongs to Lip (α,q) on $[0,\pi]$.

10.3. PROOF OF THE THEOREM. Following Sharma et al. (1977), we write

$$s_{k,p}(x)-f(x)=\frac{1}{2\pi(k+1)}\int_0^{\pi}\phi(t)\frac{\cos pt-\cos(k+p+1)t}{2\sin^2 \frac{t}{2}}dt .$$

We have

$$f(t)-t_{n,p}(t)=\frac{1}{P_n}\sum_{k=0}^{n}p_k\{f(t)-s_{k,p}(t)\}$$

$$=\frac{1}{2\pi P_n}\int_0^{\pi}\phi(t)\sum_{k=0}^{n}\frac{p_k}{(k+1)}\frac{[\cos(k+p+1)t-\cos pt]}{2\sin^2 \frac{t}{2}}dt$$

$$=\frac{1}{2\pi P_n}\left[\int_0^{1/P_n}+\int_{1/P_n}^{\pi}\right]\phi(t)\left(\sum_{k=0}^{n}\frac{-p_k}{(k+1)}\frac{\sin(k+2p+1)\frac{t}{2}\sin(k+1)\frac{t}{2}}{\sin^2 \frac{t}{2}}dt\right.$$

$$=I_1+I_2\ , \quad \text{say} .$$

Now

$$I_1=\frac{1}{2\pi P_n}\int_0^{1/P_n}\phi(t)\sum_{k=0}^{n}\frac{-p_k}{(k+1)}\frac{\sin(k+2p+1)\frac{t}{2}\sin(k+1)\frac{t}{2}}{\sin^2 \frac{t}{2}}dt$$

By Hölders inequality and the lemma, we have

$$I_1\le\frac{1}{2\pi P_n}\left\{\left(\int_0^{1/P_n}\left(\frac{t|\phi(t)|}{t^{\alpha}}\right)^p dt\right)^{1/p}\right\}$$

$$\cdot\left\{\left(\int_0^{1/P_n}\left(\frac{1}{t^{1-\alpha}}\left|\sum_{k=0}^{n}\frac{p_k}{(k+1)}\frac{\sin(k+2p+1)\frac{t}{2}\sin(k+1)\frac{t}{2}}{\sin^2 \frac{t}{2}}\right|\right)^q dt\right)^{1/q}\right\}$$

$$=O\left(\frac{1}{P_n}\right)O\left(\frac{1}{P_n}\right)O\left\{\left(\int_0^{1/P_n}\left(\frac{1}{t^{1-\alpha}}\sum_{k=0}^{n}p_k\frac{1}{t}\right)^q dt\right)^{1/q}\right\}$$

$$= O\left(\frac{1}{P_n}\right) O\left\{\left(\int_0^{1/P_n} t^{\alpha q - 2q}\right)^{1/q}\right\}$$

$$= O\left(\frac{1}{P_n}\right) O\left\{\left(\frac{1}{P_n}\right)^{\alpha - 2 + 1/q}\right\}$$

$$= O\left\{\left(\frac{1}{P_n}\right)^{\alpha - 1 + 1/q}\right\}$$

$$= O\left\{\frac{1}{(P_n)^{\alpha - 1/p}}\right\}$$

where $\frac{1}{p} + \frac{1}{q} = 1$ such that $1 \le p \le \infty$.

Also , similarly as above

$$I_2 = O\left(\frac{1}{P_n}\right)\left\{\left(\int_{1/P_n}^{\pi} \left(\frac{t^{-\delta}|\phi(t)|}{t^{\alpha}}\right)^p dt\right)^{1/p}\right\}$$

$$\cdot\left\{\left(\int_{1/P_n}^{\pi} \left|\frac{1}{t^{-\delta-\alpha}} \sum_{k=0}^{n} \frac{p_k}{(k+1)} \frac{\sin(k+2p+1)\frac{t}{2}\sin(k+1)\frac{t}{2}}{\sin^2 \frac{t}{2}}\right|^q dt\right)^{1/q}\right\}$$

where δ is finite quantity.
Now

$$I_2 = O\left(\frac{1}{P_n}\right) O\left\{\left(\int_{1/P_n}^{\pi} \left(\frac{t^{-\delta} t^{\alpha - 1/p}}{t^{\alpha}}\right)^p dt\right)^{1/p}\right\}$$

$$\cdot\left\{\left(\int_{1/P_n}^{\pi} \frac{1}{t^{-\delta-\alpha}} \sum_{k=0}^{n} \left(\frac{\left|p_k \sin(k+2p+1)\frac{t}{2}\right|}{(k+1)} \frac{(k+1)\left|\sin\frac{t}{2}\right|}{\sin^2 1/2}\right)^q dt\right)^{1/q}\right\}$$

$$= O\left(\frac{1}{P_n}\right) O\left\{\left(\frac{1}{P_n}\right)^{-\delta}\right\} O\left\{\left(\int_{1/P_n}^{\pi} \frac{t^{\alpha+\delta}}{\sin 1/2} \sum_{k=0}^{n} \left|p_k \sin(k+2p+1)\frac{t}{2}\right|^{q}\right)^{1/q}\right\}$$

$$= O\left(\frac{1}{P_n}\right) O\left\{\left(\frac{1}{P_n}\right)^{-\delta}\right\} O\left\{\left(\int_{1/P_n}^{\pi} t^{\alpha q+\delta q-2q} dt\right)^{1/q}\right\}$$

since $\{p_n\}$ is monotonic increasing, we have

$$\sum_{k=0}^{n} p_k \sin(k+2p+1)\frac{t}{2} \le p_n \sum_{k=0}^{n} \sin(k+2p+1)\frac{t}{2}$$

$$= O\left(\frac{p_n}{t}\right) \text{ [alternatively see (1)]}$$

Now

$$I_2 = O\left(\frac{1}{P_n}\right) O\left\{\left(\frac{1}{P_n}\right)^{\delta}\right\} O\left\{\left(\frac{1}{P_n}\right)^{\alpha+\delta-2+\frac{1}{q}}\right\}$$

$$= O\left\{\left(\frac{1}{P_n}\right)^{\alpha - 1 + \frac{1}{q}}\right\}$$

$$= O\left\{\frac{1}{(P_n)^{\alpha-1/p}}\right\}$$

Hence $\|f - t_{n,p}\|_p = O\left\{\frac{1}{(P_n)^{\alpha-1/p}}\right\}$

This completes the proof of theorem.

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