CHAPTER – 8

ON DOUBLE MATRIX SUMMABILITY OF DOUBLE FOURIER SERIES

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8.1 INTRODUCTION :

Harmonic and (N, p_n) summabilities of single Fourier series have been studied by a number of researchers like Iyengar (1943), Siddiqi (1948), Pati (1961), Singh (1963). In 1953 Chow for the first time studied Cesàro summability of double Fourier series. In 1958 Sharma extended the results of Chow to (H, 1, 1) summability which is weaker than (C, 1, 1) summability of double Fourier series. Working in the same direction in 1932, Hille and Tamarkin defined double Nörlund summability of double Fourier series. After this, double Nörlund summability of double Fourier series has been studied by Tripathi and Ojha (1982). But nothing seems to have been done so a in the direction of study of double Fourier series by double matrix summability methods which, as known, includes as special cases, the methods of (C, 1, 1), (H, 1, 1) and (N, p_m, q_n) summability.

In this chapter a more general result than those of Sharma, Chow, Tripathi and Ojha has been established so that their results come out as particular cases.

8.2 DEFINITIONS AND NOTATIONS :

Let f (u, v) be a function of (u, v) periodic with respect to u and with respect to v, in each case with period 2π, and summable in the square Q $(-\pi, -\pi; \pi, \pi)$. The double fourier series of the function f (u, v) is given by

$$f(u,v) \sim \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \lambda_{m,n} \big[a_{m,n} \cos mu \cos nv + b_{m,n} \sin mu \cos nv$$

$$+ c_{m,n} \cos mu \sin nv + d_{m,n} \sin mu \sin nv \big]$$

(8.2.1) $$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \lambda_{m,n} A_{m,n}(u,v).$$

where

$$\lambda_{m,n} = \begin{cases} \frac{1}{4} & , \text{ for } m = 0 , n = 0 \\ \frac{1}{2} & , \text{ for } m > 0 , n = 0 \text{ and } m = 0, n > 0 \\ 1 & , \text{ for } m > 0 , n > 0 \end{cases}$$

and

$$a_{m,n} = \frac{1}{\pi^2} \iint_Q f(u,v) \cos mu \cos nv \, du \, dv ,$$

with similar three expressions for $b_{m,n}$, $c_{m,n}$ and $d_{m,n}$; where Q denotes the fundamental square $Q(-\pi,-\pi;\pi,\pi)$. The double series $\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \lambda_{m,n} a_{m,n}$, with the sequence of (m, n)-th partial sums $\{s_{m,n}\}$ is said to be summable by double matrix summability method or summable (T, s) if $t_{m,n}$ tend to a limit s as m, n$\rightarrow \infty$, where the double matix mean $t_{m,n}$ is given by

$$t_{m,n} = \sum_{i=0}^{m}\sum_{k=0}^{n} a_{m,i} b_{n,k} s_{i,k}$$

(8.2.2) $$= \sum_{i=0}^{m}\sum_{k=0}^{n} a_{m,m-i} b_{n,n-k} s_{m-i,\, n-k} .$$

The regularity conditions of double matrix summability means are given by

$$\sum_{i=0}^{m}\sum_{k=0}^{n} a_{m,i} b_{n,k} \rightarrow 1 \quad \text{as } m,n \rightarrow \infty$$

$$\lim \sum_{k=0}^{n} |a_{m,i} b_{n,k}| = 0 , \quad \text{for } i = 1,2, \ldots\ldots\ldots\ldots\ldots ,$$

$$\lim \sum_{i=0}^{m} |a_{m,i} b_{n,k}| = 0 , \quad \text{for } i = 1,2, \ldots\ldots\ldots\ldots\ldots ,$$

$$\sum_{i=0}^{m}\sum_{k=0}^{n} \left|a_{m,i} b_{n,k}\right| \le M, \text{ a finite constant, } m, n = 0, 1, 2, \ldots\ldots\ldots\ldots .$$

Three important particular cases of double matrix summability method are

(i) (C, 1, 1) summability (Chow (1953))

if $$a_{m,i} = \frac{1}{m+1}, \ \forall\, m \ \text{ and } \ b_{n,k} = \frac{1}{n+1}, \ \forall\, n.$$

(ii) (H, 1, 1) summability (Sharma (1958))

if $$a_{m,i} = \frac{1}{(m-i+1)\log m} \ \text{ and } \ b_{n,k} = \frac{1}{(n-k+1)\log n}.$$

(iii) (N, p_m, q_n) summability mean (Hille and Tamarkin (1932))

if $$a_{m,i} = \frac{p_{m-i}}{P_m} \ \text{ and } \ b_{n,k} = \frac{q_{n-k}}{Q_n},$$

where $$P_m = \sum_{i=0}^{m} p_i \quad \text{and} \quad Q_n = \sum_{k=0}^{n} q_k.$$

For arbitrary but fixed (x, y), we write

$$\phi(u,v) = \phi(x,y;u,v)$$

$$= \frac{1}{4}\left[f(x+u,y+v) + f(x+u,y-v) + f(x-u,y+v) + f(x-u.y-v) - 4\, f(x.y)\right];$$

$$\Phi_1(u,v) = \int_0^u \int_0^v \left|\phi(s,t)\right| ds\, dt\ ;$$

$$\Phi_1(u,t) = \int_0^u \left|\phi(s,t)\right| ds\ ;$$

$$\Phi_2(s,v) = \int_0^v \left|\phi(s,t)\right| dt\ ;$$

$$\tau = \left[\frac{1}{t}\right] = \text{Integral part of } \frac{1}{t};$$

$$\sigma = \left[\frac{1}{s}\right] = \text{Integral part of } \frac{1}{s};$$

and

$$(8.2.3) \qquad K_m(u) = \frac{1}{2\pi}\sum_{i=0}^{m} a_{m,m-i}\,\frac{\sin\left(m-i+\frac{1}{2}\right)u}{\sin\frac{u}{2}};$$

$$(8.2.4) \qquad K_n(v) = \frac{1}{2\pi}\sum_{k=0}^{n} b_{n,n-k}\,\frac{\sin\left(n-k+\frac{1}{2}\right)v}{\sin\frac{v}{2}};$$

8.3 KNOWN THEOREMS. Dealing with Harmonic summability of double Fourier series, Sharma (1958) proved the following theorem :

THEOREM A . If the conditions

$$(8.3.1) \qquad \Phi(u,v) = o\left[\frac{u\,v}{\log\left(\frac{1}{u}\right)\log\left(\frac{1}{v}\right)}\right],$$

$$(8.3.2) \qquad \int_0^{\pi} \phi_1(u,t)du = o\left[\frac{u}{\log\left(\frac{1}{u}\right)}\right],$$

and

$$(8.3.3) \qquad \int_0^{\pi} \phi_2(s,v)ds = o\left[\frac{v}{\log\left(\frac{1}{v}\right)}\right],$$

hold then the double Fourier series of the function f (u, v) is summable (H,1, 1) to f (x, y) at the point (u, v) = (x, y).

This theorem is a generalization of the theorem due to Hille and Tamarkin (1932) for double Fourier series and is analogous to the theorem of Chow (1953) for smmability (C, 1,1) of double Fourier series.

Generalizing Theorem A, Tripathi and Ojha (1982) proved the following theorem:

THEOREM B. If (N, p_m, q_n) is a regular double Nörlund method defined by real, non-negative, monotonic non-increasing sequences of coefficients $\{p_m\}$ and $\{q_n\}$, such that $P_m \to \infty$ and $Q_n \to \infty$, when

(8.3.4) $\log m = O(P_m), \quad \text{as } m \to \infty$

and

(8.3.5) $\log n = O(Q_n), \quad \text{as } n \to \infty$

then, if

$$\Phi(u,v) = o\left[\frac{uv}{P_{\left[\frac{1}{u}\right]} Q_{\left[\frac{1}{v}\right]}}\right]$$

then the double Fourier series of the function $f(u, v)$ is summable (N, p_m, q_n) to the sum $f(x, y)$ at the point $(u, v) = (x, y)$.

8.4 MAIN THEOREM. A quite good amount of work are know on (C, 1, 1), (H, 1,1) and (N, p_m, q_n) summabilities of double Fourier series. But till now no work seems to have been done on double matrix summability of double Fourier series. The purpose of this chapter, is to extend the result of Chow, Sharma, Tripathi and Ojha on (C, 1, 1), (H, 1,1) and double Nörlund summability methods by a more general class of double matrix summability of double Fourier series in two ways. In fact, in this chapter, we establish the following theorem :

THEOREM. If the conditions

(8.4.1) $$|\Phi(u,v)| = \int_0^u \int_0^v |\phi(s,t)| ds\, dt$$

$$= o\left[\frac{u\alpha(1/u)}{R(1/u)} \frac{v\alpha(1/v)}{R(1/v)}\right], \text{ as } (u,v) \to +0$$

(8.4.2) $$\int_0^\pi \phi_1(u,t)dt = o\left[\frac{u\alpha(1/u)}{R(1/u)}\right]$$

and

(8.4.3) $$\int_0^\pi \phi_2(s,v)ds = o\left[\frac{v\alpha(1/v)}{R(1/v)}\right]$$

hold then the double Fourier series (8.2.1) is double matrix summable to f (x,y) at the point (u, v) = (x, y), provided

$\|T\| = (a_{m,i})$ and $\|S\| = (b_{n,k})$ be two infinite triangular matrices with $a_{m,i} \geq 0, b_{n,k} \geq 0$.

$$A_{m,\tau} = \sum_{i=0}^{\tau} a_{m,m-i} \quad , \quad B_{n,\eta} = \sum_{k=0}^{\eta} a_{n,n-k} \quad ,$$

$A_{m,m} = 1$ for each $m \geq 0$ and $B_{n,n} = 1$ for $n \geq 0$.

Let $\{a_{m,i}\}_{i=o}^{m}$ and $\{b_{n,k}\}_{k=o}^{n}$ be two real non-negative and non-decreasing sequence with respect to i and k respectively and $\alpha(t)$ be a positive monotonic non-decreasing function such that

$$\frac{\alpha(t)}{R(t)} = O(1) \text{ as } t \to \infty .$$

(8.4.4) $$\int_1^m \frac{\alpha(x)A_{m,x}}{x\,R(x)} dx = O(1) ,$$

and

(8.4.5) $$\int_1^n \frac{\alpha(y)B_{n,y}}{y\,R(y)} dy = O(1) .$$

8.5 For the proof of our theorem the following lemmas are required:

LEMMA 1. (Lal and Pratap (1999)) Let $K_m(s)$ and $K_n(t)$ be given by (8.2.3) amd (8.2.4) respectively, then

(i) $K_m(s) = O(m)$, for $o \le s \le (1/m)$

(ii) $K_n(t) = O(n)$, for $o \le t \le (1/n)$

LEMMA 2. (LaL (2000)) (i) If $(a_{m,\mu})$ is non-negative and non-decreasing with μ, then for $0 \le a \le b \le \infty$, $0 \le s \le \pi$ and any

$$\left| \sum_{\mu=a}^{b} a_{m,m-\mu} e^{i(m-\mu)s} \right| \le O(A_{m,\sigma})$$

where σ = integral part of $\frac{1}{s}$.

(ii) If $(b_{n,v})$ is non-negative and non-decreasing with v, then for $0 \le a \le b \le \infty$, $0 \le t \le \pi$ and any n,

$$\left| \sum_{v=a}^{b} a_{n,n-v} e^{i(n-v)t} \right| \le O(B_{n,\tau}),$$

where τ = integral part of $\left[\frac{1}{t}\right]$.

LEMMA 3. Let $K_m(s)$ be as given by (8.2.3), then under the condition of our theorem on $(a_{m,\mu})$

(8.5.3) $$K_m(s) = O\left(\frac{A_{m,\sigma}}{s}\right), \text{ for } 0 < \frac{1}{m} \le s \le \pi .$$

PROOF OF LEMMA 3. Since for $0 < \frac{1}{m} \le s \le \pi$, $\sin\frac{s}{2} \le \left(\frac{s}{\pi}\right)$, therefor for t > 0 and $s \le m$

We have

$$|K_m(s)| = \left| \frac{1}{2\pi} \sum_{\mu=0}^{m} a_{m,m-\mu} \frac{\sin\left(m-\mu-\frac{1}{2}\right)s}{\sin\left(\frac{s}{2}\right)} \right|$$

$$\leq \frac{1}{2\pi} \left| I_m \sum_{\mu=0}^{m} a_{m,m-\mu} \frac{e^{i\left(m-\mu-\frac{1}{2}\right)s}}{\sin\left(\frac{s}{2}\right)} \right|$$

$$= O\left(\frac{1}{s}\right)\left[\left|\sum_{\mu=0}^{m} a_{m,m-\mu} e^{i(m-\mu)s}\right| \cdot \left|e^{is/2}\right|\right]$$

$$= O\left(\frac{1}{s}\right)\left[\left|\sum_{\mu=0}^{m} a_{m,m-\mu} e^{i(m-\mu)s}\right|\right]$$

$$= O\left(\frac{A_{m,\sigma}}{s}\right), \qquad \text{by lemma 2}$$

LEMMA 4. Let $K_n(t)$ be as given by (8.2.4) , then under the condition of our theorem on $(a_{n,\nu})$,

(8.5.4) $$K_n(t) = O\left(\frac{A_{n,\tau}}{t}\right), \text{ for } o < \frac{1}{n} \leq t \leq \pi .$$

8.6 PROOF OF THE THEOREM. (i, k)-th partial sum of the series (8.2.1) at $(u,v) = (x, y)$ is given by

$$s_{i,k} - f(x,y) = \frac{1}{4\pi^2}\left[\int_0^{\pi}\int_0^{\pi} \phi(s,t) \frac{\sin\left(i+\frac{1}{2}\right)t}{\sin(\frac{s}{2})} \frac{\sin\left(k+\frac{1}{2}\right)t}{\sin(\frac{t}{2})} ds\,dt\right]$$

Then,

$$\sum_{i=0}^{m}\sum_{k=0}^{n} a_{m,i}b_{n,k}\{s_{i,k}-f(x,y)\}=\sum_{i=0}^{m}\sum_{k=0}^{n} a_{m,m-i}b_{n,n-k}\{s_{m-i,n-k}-f(x,y)\}$$

or

$$t_{m,n}-f(x,y)=\int_0^{\pi}\int_0^{\pi}\phi(s,t)K_m(s)K_n(t)ds\,dt$$

$$=\left(\int_0^{\delta}\int_0^{\xi}+\int_0^{\delta}\int_{\delta}^{\pi}+\int_{\delta}^{\pi}\int_0^{\xi}+\int_{\delta}^{\pi}\int_{\xi}^{\pi}\right)\phi(s,t)K_m(s)K_n(t)ds\cdot dt$$

$$=I_1+I_2+I_3+I_4\ \text{, say}.$$

Now considering

$$I_1=\int_0^{\delta}\int_0^{\xi}\phi(s,t)K_m(s)K_n(t)ds\,dt$$

$$=\left(\int_0^{1/m}\int_0^{1/n}+\int_{1/m}^{\delta}\int_0^{1/n}+\int_0^{1/m}\int_{1/n}^{\xi}+\int_{1/m}^{\delta}\int_{1/n}^{\xi}\right)\phi(s,t)K_m(s)K_n(t)ds\,dt$$

$$=I_{1.1}+I_{1.2}+I_{1.3}+I_{1.4}\quad\text{, say}.$$

Now, let us consider

$$I_{1.1}=\int_0^{1/m}\int_0^{1/n}\phi(s,t)K_m(s)K_n(t)ds\,dt$$

$$=O(m\,n)\int_0^{1/m}\int_0^{1/n}|\phi(s,t)|ds\,dt$$

$$=O(m\,n)o\left[\frac{\alpha(m)}{m\,R(m)}\frac{\alpha(n)}{n\,R(n)}\right],\qquad\text{by (8.4.1)}$$

$$=o(1)\ \text{, as }\ m,n\to\infty,\qquad\text{by the hypothesis of the theorem.}$$

Now

$$I_{1.2}=\int_{1/m}^{\delta}\int_0^{1/n}|\phi(s,t)|\,|K_m(s)|\,|K_n(t)|ds\,dt$$

$$\le\left[\int_0^{1/n}|K_n(t)dt|\int_{1/m}^{\delta}|\phi(s,t)|\frac{A_{m,[1/s]}}{s}ds\right],\qquad\text{by (8.5.3)}$$

$$=\left[\int_0^{1/n} O(n)dt\int_{1/m}^{\delta} |\phi(s,t)|\frac{A_{m,[1/s]}}{s}ds\right]$$ by lemma 1

$$=O(n)\left[\int_0^{1/n} dt\int_{1/m}^{\delta} |\phi(s,t)|\frac{A_{m,[1/s]}}{s}ds\right]$$

$$=O(n)\left[\int_0^{1/n} dt\left\{\left(\Phi_1(s,t)\frac{A_{m,[1/s]}}{s}\right)_{1/m}^{\delta} - \int_{1/m}^{\delta} \Phi_1(s,t)\frac{d}{ds}\left(\frac{A_{m,[1/s]}}{s}\right)ds\right\}\right]$$

$$=O(n)\left[\int_0^{1/n} \Phi_1(s,t)\left(\frac{A_{m,[1/\delta]}}{\delta}\right)dt\right]+O(n)mA_{m,m}\int_0^{1/n} \Phi_1\left(\frac{1}{m},t\right)dt$$

$$+O(n)\left[\int_0^{1/n} dt\int_{1/m}^{\delta} |\Phi_1(s,t)|\frac{d}{ds}\left(\frac{A_{m,[1/s]}}{s}\right)ds\right]$$

$$=I_{1.2.1}+I_{1.2.2}+I_{1.2.3}\,,$$ say.

Now

$$I_{1.2.1}=O(n)\int_0^{1/n} \Phi_1(\delta,t)dt$$

$$=O(n)\Phi\left(\delta,\frac{1}{n}\right)$$

$$=O(n)o\left[\frac{\delta\alpha\left(\frac{1}{\delta}\right)}{R(1/\delta)}\frac{\frac{1}{n}\alpha(n)}{R(n)}\right]$$

$$=o\left[\frac{\alpha(n)}{R(n)}\right]$$

$=o(1)$, as $m, n \to \infty$, by the hypothesis of the theorem.

Next

$$I_{1.2.2} = O(mn)\int_0^{1/n} \Phi_1\left(\frac{1}{m}, t\right)dt$$

$$= O(mn)\Phi\left(\frac{1}{m}, \frac{1}{n}\right)$$

$$= O(mn)o\left[\frac{\frac{1}{m}\alpha(m)}{R(m)} \frac{\frac{1}{n}\alpha(n)}{R(n)}\right]$$

$$= o\left[\frac{\alpha(m)}{R(m)} \frac{\alpha(n)}{R(n)}\right]$$

$= o(1)$, as $m, n \to \infty$, by the hypothesis of the theorem.

Lastly,

$$I_{1.2.3} = O(n)\left[\int_0^{1/n} dt \int_{1/m}^{\delta} \Phi_1(s,t)\frac{A_{m,[1/s]}}{s^2}ds\right] + O(n)\left[\int_0^{1/n} dt \int_{1/m}^{\delta} \frac{\Phi_1(s,t)}{s} \frac{d}{ds} A_{m,[1/s]}ds\right]$$

$= I_{1.2.3.1} + I_{1.2.3.2}$, say.

Now

$$I_{1.2.3.1} = O(n)\left[\int_0^{1/n} dt \int_{1/m}^{\delta} \Phi_1(s,t)\frac{A_{m,[1/s]}}{s^2}ds\right]$$

$$\le O(n)\left[\int_{1/m}^{\delta} \left\{\int_0^{1/n} \Phi_1(s,t)dt\right\}\frac{A_{m,[1/s]}}{s^2}ds\right]$$

$$= O(n)\left[\int_{1/m}^{\delta} \Phi(s, 1/n)\frac{A_{m,[1/s]}}{s^2}ds\right]$$

$$= O(n)\int_{1/m}^{\delta} o\left(\frac{s\alpha(1/s)}{R(1/s)} \frac{\frac{1}{n}\alpha(n)}{R(n)}\right)\frac{A_{n,[1/s]}}{s^2}ds$$

$$= o\left(\frac{\alpha(n)}{R(n)}\right)\int_{1/m}^{\delta} \frac{\alpha(1/s)}{s}\frac{A_{m,[1/s]}}{R(1/s)}ds$$

$$= o\left(\frac{\alpha(n)}{R(n)}\right)\int_{1/\delta}^{m} \frac{\alpha(x)}{xR(x)}A_{m,x}dx$$

$$= o(1) \ , \ \text{as } m,n \to \infty .$$

Next,

$$I_{1.2.3.2} = O(n)\left[\int_0^{1/n} dt\int_{1/m}^{\delta} \frac{\Phi_1(s,t)}{s}\frac{d}{ds}A_{m,[1/s]}ds\right]$$

$$\le O(n)\int_{1/m}^{\delta} \left\{\int_0^{1/n} \Phi_1(s,t)dt\right\}\frac{1}{s}\frac{d}{ds}\left(A_{m,[1/s]}\right)ds$$

$$= O(n)\int_{1/m}^{\delta} \Phi\left(s,\frac{1}{n}\right)\frac{1}{s}\frac{d}{ds}\left(A_{m,[1/s]}\right)ds$$

$$= O(n)\int_{1/m}^{\delta} o\left(\frac{s\alpha(1/s)}{R(1/s)}\frac{\frac{1}{n}\alpha(n)}{R(n)}\right)\frac{1}{s}\frac{d}{ds}\left(A_{m,[1/s]}\right)ds$$

$$= o\left(\frac{\alpha(n)}{R(n)}\right)\int_{1/m}^{\delta} \frac{\alpha(1/s)}{R(1/s)}\frac{d}{ds}\left(A_{m,[1/s]}\right)ds$$

$$= o\left(\frac{\alpha(n)}{R(n)}\right)\int_{m}^{1/\delta} \frac{\alpha(x)}{R(x)}\frac{d}{dx}\left(A_{m,x}\right)dx$$

$$= o\left(\frac{\alpha(n)}{R(n)}\right)\int_{m}^{1/\delta} O(1)\frac{d}{dx}\left(A_{m,x}\right)dx$$

$$= o\left(\frac{\alpha(n)}{R(n)}\right)\left(A_{m,x}\right)_m^{1/\delta}$$

$$= o\left(\frac{\alpha(n)}{R(n)}\right)O(1)$$

$= o(1)$, as $m, n \to \infty$, by hypothesis of the theorem.

Thus, we have

$$I_{1.2} = o(1)\ ,\ \text{as}\ m,n \to \infty$$

similarly

$$I_{1.3} = o(1)\ ,\ \text{as}\ m,n \to \infty$$

Now

$$I_{1.4} = \left[\int_{1/m}^{\delta}\int_{1/n}^{\xi} |\Phi(s,t)| \frac{A_{m,[1/s]}}{s}\frac{B_{n,[1/t]}}{t}\,dt\,ds\right]$$

$$= \frac{B_{n,\left[\frac{1}{\xi}\right]}A_{m,\left[\frac{1}{\delta}\right]}\Phi(s,\xi)}{s\xi} - \frac{mB_{n,\left[\frac{1}{\xi}\right]}A_{m,m}\Phi\left(\frac{1}{m},\xi\right)}{\xi}$$

$$-\frac{B_{n,\left[\frac{1}{\xi}\right]}}{\xi}\int_{1/m}^{\delta}\Phi(s,\xi)\frac{d}{ds}\left(\frac{A_{m,\left[\frac{1}{s}\right]}}{s}\right)ds - \frac{nB_{n,n}A_{m,\left[\frac{1}{\delta}\right]}\Phi\left(s,\frac{1}{n}\right)}{\delta}$$

$$-mn\,B_{n,n}A_{m,m}\Phi\left(\frac{1}{m},\frac{1}{n}\right) + nB_{n,n}\int_{1/m}^{\delta}\Phi\left(s,\frac{1}{n}\right)\frac{d}{ds}\left(\frac{A_{m,\left[\frac{1}{s}\right]}}{s}\right)ds$$

$$+\frac{A_{m,\left[\frac{1}{\delta}\right]}}{\delta}\int_{1/n}^{\delta}\Phi_2(s,t)\frac{d}{dt}\left(\frac{B_{n,\left[\frac{1}{t}\right]}}{t}\right)dt - mA_{m,m}\int_{1/n}^{\xi}\Phi_2\left(\frac{1}{m},t\right)\frac{d}{dt}\left(\frac{B_{n,\left[\frac{1}{t}\right]}}{t}\right)dt$$

$$-\int_{\frac{1}{m}}^{\delta}\int_{\frac{1}{n}}^{\xi}\Phi(s,t)\frac{d}{ds}\left(\frac{A_{m,\left[\frac{1}{s}\right]}}{s}\right)\frac{d}{dt}\left(\frac{B_{n,\left[\frac{1}{t}\right]}}{t}\right)dt\,ds$$

$$= I_{1.4.1} + I_{1.4.2} + I_{1.4.3} + I_{1.4.4} + I_{1.4.5} + I_{1.4.6} + I_{1.4.7} + I_{1.4.8} + I_{1.4.9}\ , \qquad \text{say.}$$

Let us consider

$$I_{1.4.1} = \frac{B_{n,\left[\frac{1}{\xi}\right]} A_{m,\left[\frac{1}{\delta}\right]} \Phi_{(\delta,\xi)}}{\delta\xi}$$

$$= \frac{B_{n,\left[\frac{1}{\xi}\right]} A_{m,\left[\frac{1}{\delta}\right]}}{\delta\xi} o \left[\frac{\delta\alpha\left(\frac{1}{\delta}\right)}{R(1/\delta)} \frac{\xi\alpha\left(\frac{1}{\xi}\right)}{R(1/\xi)} \right]$$

$$= o \left[\frac{B_{n,\left[\frac{1}{\xi}\right]}}{\xi} \frac{A_{m,\left[\frac{1}{\delta}\right]}}{\delta} \frac{\xi\delta\alpha\left(\frac{1}{\delta}\right)}{R(1/\delta)} \frac{\alpha\left(\frac{1}{\xi}\right)}{R(1/\xi)} \right]$$

$= o(1)$, as $m,n \to \infty$

Next ,

$$I_{1.4.2} = \frac{m\, B_{n,\left[\frac{1}{\xi}\right]} A_{m,m} \Phi_{\left(\frac{1}{m},\xi\right)}}{\xi}$$

$$= o \left[\frac{m\, B_{n,\left[\frac{1}{\xi}\right]}}{\xi} \left(\frac{\frac{1}{m}\alpha(m)}{R(m)} \frac{\xi\alpha(1/\xi)}{R(1/\xi)} \right) \right], (A_{m,m} = 1)$$

$$= o \left[B_{n,\left[\frac{1}{\xi}\right]} \frac{\alpha(m)}{R(m)} \frac{\alpha(1/\xi)}{R(1/\xi)} \right]$$

$= o(1)$, as $m,n \to \infty$, by hypothesis of the theorem.

Now , let us consider

$$I_{1.4.3} = -\frac{B_{n,\left[\frac{1}{\xi}\right]}}{\xi}\int_{1/m}^{\delta} \Phi(s,\xi)\frac{d}{ds}\left(\frac{A_{m,\left(\frac{1}{s}\right)}}{s}\right)ds$$

$$= o\left(\frac{B_{n,\left[\frac{1}{\xi}\right]}}{\xi}\right)\int_{1/m}^{\delta}\left(\frac{s\alpha\left(\frac{1}{s}\right)}{R\left(\frac{1}{s}\right)}\frac{\xi\alpha\left(\frac{1}{\xi}\right)}{R\left(\frac{1}{\xi}\right)}\right)\frac{d}{ds}\left(\frac{A_{m,\left[\frac{1}{s}\right]}}{s}\right)ds$$

$$= o\left(B_{n,\left[\frac{1}{\xi}\right]}\right)\int_{m}^{1/\delta} O(1)\frac{d}{dx}A_{m,x}dx$$

$$= o(1)\ , \quad \text{as} \quad m,n \to \infty$$

Thus, we get

$$I_{1.4.3} = o(1)\ , \quad \text{as} \quad m,n \to \infty$$

Similarly $I_{1.4.2}$, we get

$$I_{1.4.4} = o(1)\ , \quad \text{as} \quad m,n \to \infty$$

Further

$$I_{1.4.5} = mn\, B_{n,n} A_{m,m} \Phi\left(\frac{1}{m},\frac{1}{n}\right)$$

$$= o\left(mn\frac{\frac{1}{m}\alpha(m)}{R(m)}\frac{\frac{1}{n}\alpha(n)}{R(n)}\right)$$

$$= o\left(\frac{\alpha(m)}{R(m)}\frac{\alpha(n)}{R(n)}\right)$$

$$= o(1)\ , \quad \text{as} \quad m,n \to \infty$$

Let us consider

$$I_{1.4.6} = n\, B_{n,n} \int_{1/m}^{\delta} \Phi\left(s, \frac{1}{n}\right) \frac{1}{s} \frac{d}{ds}\left(A_{m,\left[\frac{1}{s}\right]}\right) ds$$

$$= n\, B_{n,n} \int_{1/m}^{\delta} o\left(\frac{s\alpha\left(\frac{1}{s}\right)}{R\left(\frac{1}{s}\right)} \frac{\frac{1}{n}\alpha(n)}{R(n)} \right) \frac{1}{s} \frac{d}{ds}\left(A_{m,\left[\frac{1}{s}\right]}\right) ds$$

$$= o\left[\left(\frac{\alpha(n)}{R(n)}\right) \int_{1/m}^{\delta} \frac{\alpha(1/s)}{R(1/s)} \frac{d}{ds}\left(A_{m,\left[\frac{1}{s}\right]}\right) ds \right]$$

$$= o\left[\left(\frac{\alpha(n)}{R(n)}\right) \int_{1/m}^{\delta} \left(\frac{\alpha(1/s)}{R(1/s)}\right) O(1)\, ds \right]$$

$$= o\left[\left(\frac{\alpha(n)}{R(n)}\right) \int_{m}^{1/\delta} \left(\frac{\alpha(x)}{R(x)}\right) \left(-\frac{dx}{x^2}\right) \right]$$

$$= o\left[\left(\frac{\alpha(n)}{m\, R(n)}\right) \int_{m}^{1/\delta} \left(\frac{\alpha(x)}{x\, R(x)}\right) dx \right]$$

$$= o\left[\left(\frac{\alpha(n)}{mR(n)}\right) O(1) \right]$$

$$= o(1)\ , \text{ as } m, n \to \infty .$$

Thus, we get

$$I_{1.4.6} = o(1)\ , \quad \text{as} \quad m, n \to \infty$$

As similar to $I_{1.4.3}$, we get

$$I_{1.4.7} = o(1)\ , \quad \text{as} \quad m, n \to \infty$$

As similar to $I_{1.4.6}$, we get

$$I_{1.4.8} = o(1)\ , \quad \text{as} \quad m, n \to \infty$$

Lastly ,

$$I_{1,4,9} = O\left[\int_{1/m}^{\delta}\int_{1/n}^{\xi} \Phi(s,t)\frac{1}{s}\frac{d}{ds}\left(A_{m,\left[\frac{1}{s}\right]}\right)\frac{1}{t}\frac{d}{dt}\left(B_{n,\left[\frac{1}{t}\right]}\right)dt\,ds\right.$$

$$= o\left[\int_{\frac{1}{m}}^{\delta}\int_{\frac{1}{n}}^{\xi}\left(\frac{s\alpha(1/s)}{R(1/s)}\frac{t.\alpha(1/t)}{R(1/t)}\right)\frac{1}{s}\frac{d}{ds}\left(A_{m,\left[\frac{1}{s}\right]}\right)\frac{1}{t}\frac{d}{dt}\left(B_{n,\left[\frac{1}{t}\right]}\right)dt\,ds\right]$$

$$= o\left[\int_{\frac{1}{m}}^{\delta}\int_{\frac{1}{n}}^{\xi}\left(\frac{\alpha(1/s)}{R(1/s)}\frac{\alpha(1/t)}{R(1/t)}\right)\frac{d}{ds}\left(A_{m,\left[\frac{1}{s}\right]}\right)\frac{d}{dt}\left(B_{n,\left[\frac{1}{t}\right]}\right)dt\,ds\right]$$

$$= o\left[\int_{\frac{1}{m}}^{\delta}\int_{\frac{1}{n}}^{\xi} O(1)\frac{d}{ds}\left(A_{m,\left[\frac{1}{s}\right]}\right)\frac{d}{dt}\left(B_{n,\left[\frac{1}{t}\right]}\right)dt\,ds\right]$$

$$= o\left[\left(\int_{\frac{1}{m}}^{\delta}\frac{d}{ds}\left(A_{m,\left[\frac{1}{s}\right]}\right)\right)\left(\int_{\frac{1}{n}}^{\xi}\frac{d}{dt}\left(B_{n,\left[\frac{1}{t}\right]}\right)\right)dt\,ds\right]$$

$$= o\left(A_{m,\left[\frac{1}{s}\right]}\Big|_{1/m}^{\delta}\right)\left(B_{n,\left[\frac{1}{t}\right]}\Big|_{1/n}^{\xi}\right)$$

$$= o(1)\,,\quad \text{as}\quad m,n\to\infty.$$

$$I_{1,4} = o(1)\,,\quad \text{as}\quad m,n\to\infty\,.$$

The above estimations, we get

$$I_1 = o(1)\,,\quad \text{as}\quad m,n\to\infty$$

Now $\quad m^{-1} < \delta < \pi\,,\ n^{-1} < \xi < \pi$. Then, we obtain

$$I_3 \le \left(\int_\delta^\pi \int_0^\xi\right)|\phi(s,t)|\ |K_m(s)|\ |K_n(t)|\ ds\,dt$$

$$= \int_\delta^\pi |K_m(s)|\,ds\int_0^{1/n}|\phi(s,t)|\ |K_n(t)|\,dt + \int_\delta^\pi |K_m(s)|\,ds\int_{1/n}^{\xi}|\phi(s,t)|\ |K_n(t)|\,dt$$

$$= I_{3,1} + I_{3,2}\ ,\ \text{say}\,.$$

Taking

$$I_{3.1} = \int_{\delta}^{\pi} |K_m(s)| ds \int_0^{1/n} |\phi(s,t)| \ |K_n(t)| \ dt$$

$$= O(n)\left[\int_{\delta}^{\pi} |K_m(s)| ds \int_0^{1/n} |\phi(s,t)| \ dt\right] \ , \quad \text{by lemma 1}$$

$$= O(n)\int_{\delta}^{\pi} \frac{A_{m,\delta}}{s} ds \int_0^{1/n} |\phi(s,t)| \ dt \ , \qquad \text{by (8.5.3)}$$

$$= O(n)\int_{\delta}^{\pi} \Phi_2\left(s, \frac{1}{n}\right) ds$$

$$= O(n) o\left[\frac{\frac{1}{n}\alpha(n)}{R(n)}\right]$$

$$= O(1) \ , \quad \text{as} \quad n \to \infty.$$

Further

$$I_{3.2} = \int_{\delta}^{\pi} |K_m(s)| ds \int_{1/n}^{\xi} |\phi(s,t)| \ |K_n(t)| \ dt$$

$$= \left[\int_{\delta}^{\pi} \frac{A_{m,\delta}}{s} ds \int_{1/n}^{\xi} \phi(s,t) \frac{B_{n,\tau}}{t} \ dt\right], \qquad \text{by (8.5.3) and (8.5.4)}$$

$$= O\left[\int_{\delta}^{\pi} ds \left\{\Phi_2(s,t)\frac{B_{n,\tau}}{t}\right\}_{1/n}^{\xi} - \int_{1/n}^{\xi} \Phi_2(s,t)\frac{d}{dt}\left(\frac{B_{n,\tau}}{t}\right) dt\right]$$

$$= O\left[\int_{\delta}^{\pi} ds \left\{\Phi_2(s,\xi)\frac{B_{n,\left[\frac{1}{\xi}\right]}}{\xi} - \Phi_2\left(s, \frac{1}{n}\right) n \ B_{n,n}\right\}\right] + O\left[\int_{\delta}^{\pi} ds \int_{1/n}^{\delta} \Phi_2(s,t)\frac{d}{dt}\left(\frac{B_{n,\tau}}{t}\right) dt\right]$$

$$= O\left[\int_{\delta}^{\pi} \Phi_2(s,\xi)\frac{B_{n,\left[\frac{1}{\xi}\right]}}{\xi} ds\right] + O(n)\int_{\delta}^{\pi} \Phi_2\left(s, \frac{1}{n}\right) ds + O\left[\int_{\delta}^{\pi} ds \int_{1/n}^{\delta} \Phi_2(s,t)\frac{d}{dt}\left(\frac{B_{n,\tau}}{t}\right) dt\right]$$

$$= O\left(\frac{B_{n\left[\frac{1}{\xi}\right]}}{\xi}\right)\int_{\delta}^{\pi}\Phi_2(s,\xi)ds + O(n)0\left(\frac{\alpha(n)}{nR(n)}\right) + O\left[\int_{\delta}^{\pi}ds\int_{1/n}^{\xi}\Phi_2(s,t)\frac{d}{dt}\left(\frac{B_{n,\tau}}{t}\right)dt\right]$$

$= o(1) + o(1) + o(1)$, similar to $I_{14.9}$

$= o(1)$, as $m, n \to \infty$.

Hence

$I_3 = o(1)$, as $m, n \to \infty$.

similarly, we get

$I_2 = o(1)$, as $m, n \to \infty$

By regularily of matrix summability and Riemann-Lebesgue theorem, we have

$I_4 = o(1)$, as $m, n \to \infty$

Therefore by above estimation, our theorem is completely established.

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