CHAPTER – 6

ON UNIFORM TRIANGULAR MATRIX SUMMABILITY OF LEGENDRE SERIES

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6.1 DEFINITIONS AND NOTATIONS:

Let $\sum_{n=0}^{\infty} u_n(x)$ be an infinite series with $\{s_n(x)\}$ as the sequence of its n-th partial sums. Let $(\lambda_{n,k})$ (n = 0, 1, 2, ..., k = 0, 1, ...n, $\lambda_{n,0} = 1$) be a triangular matrix of real or complex numbers. Let

(6.1.1) $$\sigma_n(x) = \sum_{k=0}^{n} \lambda_{n,k} u_k(x) = \sum_{k=0}^{n} \Delta\lambda_{n,k} s_k(x),$$

where $\Delta\lambda_{n,k} = \lambda_{n,k} - \lambda_{n,k+1}$

and

$$\Delta^2\lambda_{n,k} = \Delta\lambda_{n,k} - \Delta\lambda_{n,k+1}$$

If

(6.1.2) $$\sigma_n(x) \to s(x), \text{ as } n \to \infty,$$

then we say that the series $\sum u_n(x)$ is summable $(\wedge)$ to s (x) at a point x.

If

(6.1.3) $$\sigma_n(x) - s(x) = o(1), \text{ as } n \to \infty,$$

uniformly in a set E,

then we say that the series $\sum u_n(x)$ is summable $(\wedge)$ uniformly in set E to the sum $s(x)$.

This chapter in the form of research paper has been published in Acta Ciencia Indica, Vol. XXXIIM, No. 4, 1443-1450 (2006), (India)

In particular, if

$$\Delta\lambda_{n,k} = \begin{cases} [(n+1-k)\log n]^{-1} & , k \le n \\ 0 & , k > n, \end{cases} \tag{6.1.4}$$

then $\sigma_n(x)$ defined by (6.1.1) is same as the harmonic mean $\left(N, \frac{1}{n+1}\right)$ of the sequence $\{s_n(x)\}$.

6.2 The Legendre series associated with a Lebesgue integrable function $f(x)$ in the range $(-1,1)$ is given by

$$f(x) \sim \sum_{n=0}^{\infty} a_n P_n(x) \ , \tag{6.2.1}$$

where

$$a_n = \left(n + \frac{1}{2}\right) \int_{-1}^{1} f(x)\ P_n(x)\ dx \tag{6.2.2}$$

and the n-th Legendre polynomial $P_n(x)$ is defined by the generating function

$$\frac{1}{\sqrt{1-2xz+z^2}} = \sum_{n=0}^{\infty} P_n(x)\ z^n \tag{6.2.3}$$

We use the following notations :

$$\psi(t) = \psi_\theta(t) = f\{\cos(\theta - t)\} - f(\cos\theta) \tag{6.2.4}$$

and

$$N_n(t) = \sum_{k=0}^{n} \Delta\lambda_{n,k} \frac{\sin(k+1)t}{\sin(\frac{t}{2})} \ . \tag{6.2.5}$$

6.3 Dwivedi (1970) has established the following theorem on uniform harmonic summability of Legendre series.

THEOREM A. If

$$\int_0^t |f(x \pm u) - f(x)| du = o\left[\frac{t}{\log\left(\frac{1}{t}\right)}\right], \text{ as } t \to +0 \tag{6.3.1}$$

uniformly in a set E defined in interval $(-1,1)$ in which $f(x)$ is bounded, then the series (6.2.1) is summable by harmonic means uniformly in E to the sum $f(x)$.

In the present chapter, we propose to extend the above result for uniform triangular matrix summability of Legendre series by proving the following;

THEOREM. If

$$(6.3.2) \qquad \int_t^\eta \frac{|\psi(u)|}{u^2}\,du = o\left[\frac{\lambda(1/t)}{t\,P_{(1/t)}}\right], \text{ as } t \to +0 ,$$

where $0 \le t < \eta < \pi$ is fixed, uniformly in a set E defined in the interval $(-1,1)$ in which $f(x)$ is bounded and $\lambda(t)$ is a positive non-decreasing function of t, such that $P_n \to \infty$ and

$$(6.3.3) \qquad \lambda(n)\log n = O(P_n), \quad \text{as } n \to \infty$$

then the series (6.2.1) is summable $(\wedge)$ uniformly in a set E to the sum $f(x)$.

NOTE : It is worth noticing that our condition (6.3.2) is less stronger than the condition (6.3.1) in theorem A in the following sense.

Following on the lines Foà (1943), it may be easily proved that under condition (6.3.1), we have

$$(6.3.4) \qquad \int_0^t |f\{\cos(\theta - y)\} - f(\cos\theta)|\,dy = o\left[\frac{t\,\lambda(1/t)}{P_{(1/t)}}\right], \text{ as } t \to 0$$

where $x = \cos\theta$, $x + u = \cos\phi$, $\theta - \phi = y$; so that in view of (6.2.4)

$$(6.3.5) \qquad \int_0^t |\psi(y)|\,dy = o\left[\frac{t\lambda(1/t)}{P_{(1/t)}}\right], \text{ as } t \to +0$$

Now, it can be easily seen, as under, that (6.3.2) implies (6.3.5) on integrating by parts, we have

$$\int_0^t |\psi(y)|dy = \int_0^t y^2 \frac{|\psi(y)|}{y^2} dy$$

$$= y^2 \int_0^t \frac{|\psi(y)|}{y^2} dy - 2\int_0^t y\left[\int \frac{|\psi(y)|}{y^2} dy\right] dy$$

$$= \left[y^2 o\left(\frac{\lambda(1/y)}{y\,P_{(1/y)}}\right)\right]_0^t - 2\int_0^t y\, o\left(\frac{\lambda(1/y)}{y\,P_{(1/y)}}\right) dy$$

$$= o\left(\frac{y\lambda(1/y)}{P_{(1/y)}}\right)_0^t - 2\int_0^t o\left(\frac{\lambda(1/y)}{P_{(1/y)}}\right) dy$$

$$= o\left(\frac{t\,\lambda(1/t)}{P_{(1/t)}}\right), \quad \text{as } t \to +0$$

6.4 In due course of the proof of our theorem, we shall use the following Lemmas :

LEMMA 1.

$$(6.4.1) \qquad \sum_{\nu=0}^{n} (2\nu+1)P_\nu(x)P_\nu(y) = \frac{(n+1)P_{n+1}(y)P_n(x) - P_{n+1}(x)P_n(y)}{(y-x)}$$

This identity is known as Christoffel's formula of summation.

LEMMA 2. If $\{\Delta\lambda_{n,k}\}_{k=0}^{n}$ is a non-negative and non-decreasing sequence with respect to k, then for $0 \le a \le b \le \infty$, $0 < t \le \pi$ and for every n,

$$(6.4.2) \qquad \left|\sum_{k=a}^{b} \Delta\lambda_{n,n-k} e^{i(n-k)t}\right| = O\left[\frac{1}{t}\Delta\lambda_{n,n-\tau}\right],$$

where τ is the integral part of $\frac{1}{t}$.

LEMMA 3. If $\{\Delta\lambda_{n,k}\}_{k=0}^{n}$ is a non-negative and non-decreasing sequence with respect to k, such that

$$\sum_{k=0}^{n} \Delta\lambda_{n,k} = 1 ,$$

then ,as $n \to \infty$,

$$\Delta\lambda_{n,k} = O\left(\frac{1}{n-k+1}\right) ,$$

uniformly for all $k \le n$, so that we get

$$(6.4.3) \qquad \Delta\lambda_{n,0} = O\left(\frac{1}{n}\right) .$$

LEMMA 4. If

$$N_n(t) = \sum_{k=0}^{n} \Delta\lambda_{n,k} \frac{\sin(k+1)t}{\sin(\frac{1}{2})} ,$$

then

$$|N_n(t)| = O(n) \quad , \quad \text{as} \quad n \to \infty$$

uniformly in $0 < t \le \frac{1}{n}$.

PROOF OF LEMMA 4. We have

$$|N_n(t)| = \left| \sum_{k=0}^{n} \Delta\lambda_{n,k} \frac{\sin(k+1)t}{\sin(\frac{1}{2})} \right|$$

$$= O\left| \sum_{k=0}^{n} |\Delta\lambda_{n,k}| (k+1) \right|$$

Now , using Abel's transformation, we get

$$|N_n(t)| = O\left[\sum_{k=0}^{n-1} |k+1-k-2| \sum_{v=0}^{k} |\Delta \lambda_{n,v}| + (n+1) \sum_{k=0}^{n} |\Delta \lambda_{n,k}| \right]$$

$$= O\left[\sum_{k=0}^{n-1} \left\{ \sum_{v=0}^{k} |\Delta \lambda_{n,v}| \right\} + (n+1) \sum_{k=0}^{n} |\Delta \lambda_{n,k}| \right]$$

By the regularity condition of $(\wedge)$ summability, there exists a constant M, such that

$$\sum_{k=0}^{\infty} \left|\Delta \lambda_{n,k}\right| < M \text{ , for every n.}$$

Therefore, we have

$$\left|N_n(t)\right| = O\left[Mn + (n+1)M\right]$$

(6.4.4) $\qquad = O(n)$, as $\quad n \to \infty$,

uniformly in $0 < t \le 1/n$.

6.5 PROOF OF THE THEOREM. The n-th partial sum of the series (6.2.1) is

$$s_n(x) = \sum_{v=0}^{n} a_v P_v(x)$$

$$= \sum_{v=0}^{n} \frac{(2v+1)}{2} \int_{-1}^{1} f(y) P_v(y) P_v(x)\, dy , \qquad \text{by (6.2.2)}$$

$$= \frac{(n+1)}{2} \int_{-1}^{1} \frac{P_{n+1}(y)P_n(x) - P_{n+1}(x)P_n(y)}{y-x} f(y)\, dy \text{ , by (6.4.1)}$$

putting $f(y) = 1$, it can be easily seen that

$$1 = \frac{(n+1)}{2} \int_{-1}^{1} \frac{P_{n+1}(y)P_n(x) - P_{n+1}(x)P_n(y)}{y-x} dy$$

therefore

$$s_n(x) - f(x) = \frac{n+1}{2} \int_{-1}^{1} \left[f(y) - f(x)\right] \frac{P_{n+1}(y)P_n(x) - P_{n+1}(x)P_n(y)}{y-x} dy$$

Let us take a positive number $s < 1$ and consider it as the sum of two other positive number α and β . Let δ be another positive number such that $0 < \delta < \alpha$, αx and $\alpha x'$ be two continuous functions of x within (-1, 1) which lie within the limits $\delta \le \alpha x \le \alpha$, $\delta \le \alpha x' \le \alpha$.

Therfore, for $-1 + s \le x \le 1 - s$, we have

$$s_n(x) - f(x) = \frac{(n+1)}{2}\left[\int_{-1}^{x-\alpha x} + \int_{x-\alpha x}^{x+\alpha x'} + \int_{x+\alpha x'}^{1}\right]\left[f(y) - f(x)\right]\frac{P_{n+1}(y)P_n(x) - P_n(y)P_{n+1}(x)}{y-x}dy$$

(6.5.1) $= A_n(x) + B_n(x) + C_n(x)$, say .

Hobson (1909) has shown that uniformly for $-1+s \le x \le 1-s$

(6.5.2) $$\begin{cases} \lim_{n\to\infty} A_n(x) = 0 \\ \lim_{n\to\infty} C_n(x) = 0 \end{cases}$$

Now, we suppose that $x = \cos\theta$, $y = \cos\phi$, $0 < \theta < \pi$, $0 < \phi < \pi$,

$$1 - \beta = \cos\rho , \quad 1 - (\alpha + \beta) = 1 - s = \cos(\rho + \sigma), \quad 1 < \rho < \frac{\pi}{2} \;, 0 < \sigma , \; \rho + \sigma < \frac{\pi}{2} \; .$$

Thus, if η denotes the minimum of $[\text{arc } \cos u - \text{arc} \cos (u + \alpha)]$

for u in $(-1, 1-\alpha)$, we have on the lines of Sansone (1959) .

$$B_n(\cos\theta) = \left(\frac{n+1}{2}\right)\int_{\theta-\eta}^{\theta+\eta}\left[f(\cos\phi) - f(\cos\theta)\right]\cdot$$

$$\frac{P_{n+1}(\cos\phi)P_n(\cos\theta) - P_{n+1}(\cos\theta)P_n(\cos\phi)}{\cos\phi - \cos\theta}\sin\phi \, d\phi$$

in which $(\rho + \sigma) \le \theta \le \pi - (\rho + \sigma)$, $0 < \eta \le \sigma$. With successive transformation, we get

(6.5.3) $B_n(\cos\theta) = D_n(\theta) + E_n(\theta)$,

where $$D_n(\theta) = \frac{1}{2\pi\sqrt{\sin\theta}}\int_{\theta-\eta}^{\theta+\eta}\frac{f(\cos\phi) - f(\cos\theta)}{\sin\frac{1}{2}(\theta - \phi)} \cdot \sin\{(n+1)(\theta - \phi)\}\sqrt{\sin\phi} \;\; d\phi$$

and obviously on the line of Sansone (1959)

$$E_n(\theta) = o(1) , \quad \text{as} \quad n \to \infty ,$$

uniformly where x lies within (-1+s, 1-s) i.e. in the set E.

putting $\theta - \phi = t$, we get

(6.5.4) $$D_n(\theta)=\frac{1}{\pi\sqrt{\sin\theta}}\int_0^{\eta}\left[f\{\cos(\theta-t)\}-f(\cos\theta)\right]\cdot\frac{\sin(n+1)t}{\sin(\frac{t}{2})}\sqrt{\sin(\theta-t)}\;dt$$

So , we get from (6.5.1) to (6.5.4) ,

$$s_n(x)-f(x)=\frac{1}{\pi\sqrt{\sin\theta}}\int_0^{\eta}\left[f\{\cos(\theta-t)\}-f(\cos\theta)\right]$$

$$\frac{\sin(n+1)t}{\sin(\frac{t}{2})}\sqrt{\sin(\theta-t)}\;dt+o(1)$$

$$=O\left[\int_0^{\eta}\{f(\cos(\theta-t))-f(\cos\theta)\}\frac{\sin(n+1)t}{\sin(\frac{t}{2})}\;dt\right]$$

(6.5.5) $$=O\left[\int_0^{\eta}\psi(t)\frac{\sin(n+1)t}{\sin(\frac{t}{2})}\;dt\right]+o(1)\,,$$

uniformly in E.

Now, if $\sigma_n(x)$ be the $(\wedge)$- mean of the sequence $\{s_n(x)\}$ of partial sums of the series (6.2.1) then by the application of (6.1.1), we have

$$\sigma_n(x)-f(x)=\sum_{k=0}^{n}\Delta\lambda_{n,k}\left[s_k(x)-f(x)\right]$$

$$=O\left[\int_0^{\eta}\left\{\sum_{k=0}^{n}\Delta\lambda_{n,k}\frac{\sin(k+1)t}{\sin(\frac{t}{2})}\right\}\psi(t)dt\right]+o(1)$$

$$=O\left[\int_0^{\eta}|\psi(t)|\,|N_n(t)|\;dt\right]+o(1)$$

$$=O(n)\left[\left\{\int_0^{1/n}+\int_{1/n}^{\eta}\right\}|\psi(t)|\,|N_n(t)|\;dt\right]+o(1)$$

(6.5.6) $$=O(I_1)+O(I_2)+o(1)\,,$$

uniformly in E.

Now , the theorem will be established, if we show that

(6.5.7) $\quad I_1 = o(1)$

and $\quad I_2 = o(1) \quad , \quad$ as $\; n \to \infty$

uniformly in E.

Let us first consider I_1 . We have by (6.4.4.)

$$I_1 = O(n)\left[\int_0^{1/n} |\psi(t)| \; dt\right]$$

$$= O(n)\left[o\left(\frac{\lambda(n)}{nP_n}\right)\right]$$

$$= o\left(\frac{\lambda(n)}{P_n}\right)$$

(6.5.8) $\quad = o(1)$, as $\; n \to \infty$

Next, considering I_2 , we have

$$I_2 = \int_{1/n}^{\eta} |\psi(t)| \, |N_n(t)| dt$$

$$= \int_{1/n}^{\eta} |\psi(t)| \left| \sum_{k=0}^{n} \Delta\lambda_{n,k} \frac{\sin(k+1)t}{\sin(1/2)} \right| dt$$

$$= \int_{1/n}^{\eta} |\psi(t)| \left| \sum_{k=0}^{n} \Delta\lambda_{n,n-k} \frac{\sin(n-k+1)t}{\sin(1/2)} \right| dt$$

$$\leq \int_{1/n}^{\eta} \frac{|\psi(t)|}{t} \left| I_m \sum_{k=0}^{n} \Delta\lambda_{n,n-k} e^{i(n-k+1)t} \right| dt$$

$$= O\left[\int_{1/n}^{\eta} \frac{|\psi(t)|}{t^2} \sum_{k=0}^{n} \Delta\lambda_{n,n-\tau} dt\right], \qquad \text{by (6.4.2)}$$

$$= O\left[\int_{1/n}^{\eta} \frac{|\psi(t)|}{t^2} \Delta\lambda_{n,0} dt\right]$$

$$= O\left(\frac{1}{n}\right)\int_{1/n}^{\pi} \frac{|\psi(t)|}{t^2}\,dt \quad ,$$ by (6.4.3)

$$= O\left(\frac{1}{n}\right) o\left(\frac{n\ \lambda(n)}{P_n}\right)$$

$$= o\left(\frac{\lambda(n)}{P_n}\right)$$

(6.5.9) $$= o(1) \quad , \text{ as } n \to \infty$$

uniformaly in E .

Now combining (6.5.6), (6.5.7), (6.5.8) and (6.5.9) , we get the required result.

This completes the proof of the theorem.

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