CHAPTER – 5

ON ALMOST EULER SUMMABILITY OF A FOURIER SERIES

ON ALMOST EULER SUMMABILITY OF A FOURIER SERIES

5.1 DEFINITIONS AND NOTATIONS :

Let $\sum a_n$ be an infinite series with the sequence $\{s_n\}$ of its partial sums and let $q > 0$. A bounded sequence $\{s_n\}$ is said to be almost convergent to a finite limit s if

(5.1.1) $$\lim_{n\to\infty} \frac{1}{n+1}\sum_{v=m}^{n+m} s_v = s \ ,$$

uniformly with respect to m. (Lorentz, 1948).

We say the series $\sum a_n$ is almost summable (E, q) , q > o to s if

(5.1.2) $$\lim_{n\to\infty} t_{n,m} = \lim_{n\to\infty} (q+1)^{-n}\sum_{k=0}^{n} \binom{n}{k} q^{n-k} s_{k,m} = s \ .$$

uniformly with respect to m, where

(5.1.3) $$s_{k,m} = \frac{1}{k+1}\sum_{r=m}^{k+m} s_r$$

Let $f(t)$ be a 2π periodic and Lebesgue integrable function of t in the interval $(-\pi, \pi)$. Then the Fourier series of the function $f(t)$ is given by

$$f(t) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$$

(5.1.4) $$= \sum_{n=0}^{\infty} A_n(t)$$

Let us write

(5.1.5) $$\Phi(t) = f(x+t) + f(x-t) - 2\, f(x)$$

This chapter in the form of research paper has been published in Indian Academy of Mathematics, JIAM Vol. 30. No. 1 (2008).

(5.1.6) $$E_{n,m}(t) = (q+1)^{-n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \frac{\cos mt - \cos(k+m+1)t}{2(k+1)\sin^2\left(\frac{t}{2}\right)}$$

(5.1.7) $$\tau = \left[\frac{1}{t}\right], \text{ the integral part of } \frac{1}{t}.$$

5.2 KNOWN RESULT :

Sharma, Dixit and Shukla (1977) have established the following theorem on almost Borel summability of Fourier series.

THEOREM A. If

(5.2.1) $$\int_0^t |\phi(u)| du = o\left[\frac{t}{\left(\log\frac{1}{t}\right)^{\delta}}\right], \quad \text{as } t \to +0,$$

and

(5.2.2) $$\int_{1/(r+m+1)}^{1/(r+m+1)^{\delta/2}} \frac{|\phi(u)|}{u} du = o(1), \quad \text{as } r \to \infty,$$

where $0 < \delta < 1$, uniformly with respect to m, then the Fourier series (5.1.4) is almost summable (B) to f(x) at the point t = x.

5.3 MAIN RESULT :

In the present chapter, our object is to prove the following result :

THEOREM. Let ξ (t) and K (t) be two positive functions of t such that $\xi(t)$. K(t) and $\frac{t\cdot\xi(t)}{K(t)}$ increase monotonically with t and let $\{p_n\}$ be a non-negative, monotonic non-increasing sequence of real constats with P_n as its n-th partial sum, such that $P_n \to \infty$ as $n \to \infty$. If

(5.3.1) $$\int_0^t |\phi(u)| du = o\left[\frac{t\,\xi(1/t)}{\{K(P_\tau)\}^{\delta}}\right], \quad \text{as } t \to +0$$

and

(5.3.2) $$\int_{1/(n+m)}^{1/(n+m)^{\delta}} \frac{|\phi(u)|}{u} du = o(1) \ , \ \text{as } n \to \infty \, .$$

where $0 < \delta < 1$, uniformly with respect to m, then the series (5.1.4) is almost (E,q) summable to f(x) at the point $t = x$, provided that

(5.3.3) $$\xi(n)\log n = O\left[\{K(P_n)\}^{\delta}\right]$$

5.4 For the proof of our theorem, we shall use the following lemmas:

LEMMA. If

$$E_{n,m}(t) = (q+1)^{-n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \frac{\cos mt - \cos(k+m+1)t}{2(k+1)\sin^2(t/2)},$$

then

$$E_{n,m}(t) = \begin{cases} O(n+m) & , \text{ for } 0 < t < \dfrac{1}{(n+m)} \\ O(1/t) & , \text{ for } \dfrac{1}{(n+m)} < t < \pi \end{cases}.$$

PROOF OF THE LEMMA. We have

$$E_{n,m}(t) = (q+1)^{-n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \frac{\cos mt - \cos(k+m+1)t}{2(k+1)\sin^2(t/2)}$$

$$= (q+1)^{-n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \frac{\sin\left(m + \frac{k+1}{2}\right)t \cdot \sin\left(\frac{k+1}{2}\right)t}{(k+1)\sin^2(t/2)}$$

$$= O(n+m) \ , \ \text{for } 0 < t < \frac{1}{n+m} \, .$$

similarly, on expanding sine and cosine in power of t, we get

$$E_{n,m}(t) = O(1/t) \ , \ \text{for } 1/n < t < \pi \ .$$

5.5 PROOF OF THE THEOREM.

The n-th partial sum of the series (5.1.4) is given by

$$s_k(x) - f(x) = \frac{1}{2\pi}\int_0^{\pi} \phi(t)\frac{\sin\left(k+\frac{1}{2}\right)t}{\sin(t/2)}dt$$

so that

$$s_{k,m} - f(x) = \frac{1}{(k+1)}\sum_{r=m}^{k+m} \{s_r - f(x)\}$$

(5.5.1) $$= \frac{1}{2\pi(k+1)}\int_0^{\pi} \phi(t)\frac{\cos mt - \cos(k+m+1)t}{2\sin^2(t/2)}dt$$

Now, by (5.1.2), we have

$$t_{n,m} - f(x) = (q+1)^{-n}\frac{1}{2\pi}\int_0^{\pi}\phi(t)\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\cdot\frac{\cos mt - \cos(k+m+1)t}{2(k+1)\quad \sin^2(t/2)}dt$$

$$= \frac{1}{2\pi}\int_0^{\pi}\phi(t)E_{n,m}(t)dt$$

$$= \frac{1}{2\pi}\left[\int_0^{1/(n+m)} + \int_{1/(n+m)}^{1/(n+m)^{\delta}} + \int_{1/(n+m)^{\delta}}^{\pi}\right]\phi(t)E_{n,m}(t)dt$$

(5.5.2) $$= I_1 + I_2 + I_3 \qquad \text{, say}.$$

Let us first consider I_1. Now,

$$I_1 = O\left[\int_0^{1/n+m} |\phi(t)|\,|E_{n,m}(t)|dt\right]$$

$$= O(n+m)\int_0^{1/n+m} |\phi(t)|\,dt$$

$$= O(n+m)o\left[\frac{\xi(n+m)}{(n+m)\{K(P_{n+m})\}^{\delta}}\right]$$

$$= o\left[\frac{\xi(n+m)}{\{K(P_{n+m})\}^{\delta}}\right]$$

(5.5.3) $= o(1)$, as $n \to \infty$, by (5.5.3)

uniformly with respect to m, using the fact that $np_n \le P_n$ by the condition on $\{p_n\}$.

Next, consider I_2, we see that

$$I_2 = O\left[\int_{1/(n+m)}^{1/(n+m)^{\delta}} \frac{|\phi(t)|}{t} dt\right]$$

(5.5.4) $= o(1)$, as $n \to \infty$, by (5.3.2)

uniformly with respect to m.

Lastly, we have

$$I_3 = \frac{1}{2\pi}\int_{1/(n+m)^{\delta}}^{\pi} \phi(t)(q+1)^{-n}\sum_{k=0}^{n}\binom{n}{k} q^{n-k}\frac{\cos mt - \cos(k+m+1)t}{2(k+1)\ \sin^2(t/2)} dt$$

$$= \frac{1}{2\pi}\int_{1/(n+m)^{\delta}}^{\pi} \phi(t)(q+1)^{-n}\sum_{k=0}^{n}\binom{n}{k} q^{n-k}\frac{\cos mt}{2(k+1)\ \sin^2(t/2)} dt$$

$$- \frac{1}{2\pi}\int_{1/(n+m)^{\delta}}^{\pi} \phi(t)(q+1)^{-n}\sum_{k=0}^{n}\binom{n}{k} q^{n-k}\frac{\cos(k+m+1)t}{2(k+1)\ \sin^2(t/2)} dt$$

(5.5.5) $= I_{3.1} - I_{3.2}$, say.

Now, using second mean value theorem,

$$I_{3.1} \le (q+1)^{-n}\sum_{k=0}^{n}\binom{n}{k} q^{n-k}\frac{1}{2\sin^2\left\{\frac{1}{2(n+m)^{\delta}}\right\}}\cdot\int_{1/(n+m)^{\delta}}^{\epsilon} \phi(t)\cos mt\, dt$$

where $\frac{1}{(n+m)^{\delta}} < \epsilon < \pi$

(5.5.6) $= o(1)$, as $n \to \infty$,

uniformly with respect to m.

similarly,

$$I_{3.2} = (q+1)^{-n}\sum_{k=0}^{n}\binom{n}{k}\int_{1/(n+m)^{\delta}}^{\pi} \phi(t)\, q^{n-k}\, \frac{\cos(k+m+1)t}{2(k+1)\,\sin^{2}(t/2)}\,dt$$

$$\leq \frac{1}{2\sin^{2}\left\{\dfrac{1}{2(n+m)^{\delta}}\right\}}\int_{1/(n+m)^{\delta}}^{\pi} |\phi(t)|\,dt$$

(5.5.7) $$= o(1)\,,\ \text{as } n\to\infty\,,$$

uniformly with respect to m.

Using (5.5.6) and (5.5.7) in (5.5.5), we get

$I_3 = o(1)$, as $n\to\infty$,

uniformly with respect to m.

This completes the proof of our theorem.

❖❖❖