CHAPTER 4

APPROXIMATION BY CERTAIN
BASKAKOV KANTOROVICH OPERATORS

4.1. INTRODUCTION

Miheşan (1998) proposed generalization of well known Baskakov operators, which is based on the negative binomial distribution. Motivated by this generalization of the Baskakov operators very recently, Gupta and Singh (2016) considered for $a \geq 0$, the following operators depending on certain parameter $\alpha \geq 0$

$$V_n^{a,\alpha} (x) = \sum_{k=0}^{\infty} b^{a,\alpha}_{n,k} (x) f \left( \frac{k}{n} \right),$$

where the generalized basis function is given by

$$b^{a,\alpha}_{n,k} (x) = \frac{e^{-a\alpha(1+\alpha)}}{k!} \sum_{i=0}^{k} \binom{k}{i} (n\alpha)^i a^{k-i} \frac{(\alpha x)^k}{(1+\alpha x)^{a+k}}.$$  

It can be observed that $\sum_{k=0}^{\infty} b^{a,\alpha}_{n,k} (x) = 1$. For the special value $a = 0, \alpha = 1$ the above operators become Baskakov operators and $a = 0, \alpha \to 0$ the above operators reduce to the classical Szász-Mirakyan operators. These operators preserve only the constant functions, but for special value $a = 0$ these preserves linear functions also.

Considering the basis functions $b^{a,1}_{n,k} (x)$, Erençin (2011) studied the modified form of the operators discussed by Gupta (1994). Recently with the basis function $b^{a,\alpha}_{n,k} (x)$ Gupta and Singh (2016) proposed the Durrmeyer type operators and established some direct results in simultaneous approximation. We now propose the Kantorovich type integral modification of the operators defined in Eqn. (4.1) in the following way:

$$K_n^{a,\alpha} (f, x) = \int_{k/n}^{(k+1)/n} f(t) dt$$  

(4.3)
where \( b_{n,k}^{x,a} (x) \) is as defined in Eqn. (4.2). It is observed that the operators defined by Eqn. (4.3) produce rational functions. Also the special cases for \( a = 0 \) of operators defined by Eqn. (4.3) provide Baskakov-Kantorovich and Szász-Kantorovich operators when \( \alpha = 1 \) and \( \alpha \to 0 \) respectively.

In the present chapter, we discuss some direct estimates for the operators \( K_{n}^{x,a} (f, x) \), which includes asymptotic expansion and error estimation formula in terms of second order of the modulus of continuity by means of Steklov mean. The last segment of the chapter considers the functions with derivative of bounded variation and obtain the rate of convergence for the defined operators.

4.2. PRELIMINARIES

In order to prove direct estimates, we need the following basic results.

**Lemma 4.2.1.** For \( m \in N^0, a \geq 0 \), if we define
\[
T_{n,m}^{a,a} (x) = \sum_{k=0}^{m} b_{n,k}^{a,a} (x) \left( \frac{k}{n} \right)^m,
\]
then for \( m \geq 2 \) the following recurrence relation holds:
\[
T_{n,m+1}^{a,a} (x) = \frac{x(1+\alpha x)}{n} [T_{n,m}^{a,a} (x)] + \left[ \frac{a \alpha x}{n(1+\alpha x)} + x \right] T_{n,m}^{a,a} (x).
\]
In particular
\[
T_{n,0}^{a,a} (x) = 1, T_{n,1}^{a,a} (x) = x + \frac{a \alpha x}{n(1+\alpha x)}
\]
and
\[
T_{n,2}^{a,a} (x) = x^2 + \frac{2a \alpha x^2}{n(1+\alpha x)} + \frac{a^2 \alpha^2 x^2}{n^2(1+\alpha x)^2} + \frac{a \alpha x}{n^2(1+\alpha x)} + \frac{x(1+\alpha x)}{n}.
\]

**Proof:** From Eqn. (4.2), the result is obvious for \( m=0 \) and for \( m=1 \), we proceed as follows: Starting from the relation
\[
(1-z)^{-\alpha} e^{\alpha x} = \sum_{k=0}^{\infty} \left[ \sum_{i=0}^{k} \binom{k}{i} (x) a^{k-i} \right] \frac{z^k}{k!} \tag{4.4}
\]
Differentiating it w.r.t \( z \), we have
\[ x(1 - z)^{-x} e^{az} + a(1 - z)^{-x} e^{az} = \sum_{k=0}^{\infty} \left[ \sum_{i=0}^{k} \binom{k}{i} (x)^i a^{k-i} \right] \frac{z^{k-1}}{k!} \]

\[ \{x(1 - z)^{-1} e^{az} + a\} (1 - z)^{-x} e^{az} = \sum_{k=0}^{\infty} \left[ \sum_{i=0}^{k} \binom{k}{i} (x)^i a^{k-i} \right] \frac{z^{k-1}}{k!} \]

\[ x(1 - z)^{-1} e^{az} + a = (1 - z)^{1-x} e^{az} \frac{n}{z} \sum_{k=0}^{\infty} \left[ \sum_{i=0}^{k} \binom{k}{i} (x)^i a^{k-i} \right] \frac{z^{k-1}}{n k!} \]

Now replace \( x \) with \( n/\alpha \) and \( z \) with \( \alpha x/(1+\alpha x) \), we obtain

\[ x + \frac{\alpha \alpha x}{n(1+\alpha x)} = \sum_{k=0}^{\infty} e^{-a\alpha x/(1+\alpha x)} \left[ \sum_{i=0}^{k} \binom{k}{i} (n/\alpha)^i a^{k-i} \right] \frac{k}{n} \frac{(\alpha x)^k}{(1+\alpha x)^{n/\alpha+k}}. \]

From Eqn. (4.2), we have

\[ x + \frac{\alpha \alpha x}{n(1+\alpha x)} = \sum_{k=0}^{\infty} b_{n,k}^{a,\alpha} (x) \left( \frac{k}{n} \right) = T_{n,1}^{a,\alpha} (x). \]

Similarly Eqn. (4.2) and second derivative of Eqn. (4.4) gives \( T_{n,2}^{a,\alpha} (x) \).

To find the recurrence relation differentiating \( T_{n,m}^{a,\alpha} (x) \) w.r.t. \( x \), we obtain

\[ [T_{n,m}^{a,\alpha} (x)] = \sum_{k=0}^{\infty} \frac{d}{dx} \left( b_{n,k}^{a,\alpha} (x) \right) \left( \frac{k}{n} \right)^m \]

\[ = \sum_{k=0}^{\infty} \frac{e^{-a\alpha x/(1+\alpha x)}}{k!} \sum_{i=0}^{k} \binom{k}{i} (n/\alpha)^i a^{k-i} \left( \frac{k\alpha(\alpha x)^{k-1}}{(1+\alpha x)^{n/\alpha+k}} - \left( \frac{n}{\alpha} + k \right) \frac{\alpha(\alpha x)^k}{(1+\alpha x)^{n/\alpha+k+1}} \right) \left( \frac{k}{n} \right)^m \]

\[ - \sum_{k=0}^{\infty} \frac{e^{-a\alpha x/(1+\alpha x)}}{k!} \frac{\alpha \alpha x}{(1+\alpha x)^{n/\alpha+k}} \sum_{i=0}^{k} \binom{k}{i} (n/\alpha)^i a^{k-i} \left( \frac{\alpha(\alpha x)^{k-1}}{(1+\alpha x)^{n/\alpha+k+1}} \right) \left( \frac{k}{n} \right)^m \]

\[ = \frac{\alpha \alpha}{(1+\alpha x)^2} T_{n,m}^{a,\alpha} (x) + \frac{n}{x} T_{n,m+1}^{a,\alpha} (x) - \frac{n}{1+\alpha x} T_{n,m}^{a,\alpha} (x) - \frac{n \alpha}{1+\alpha x} T_{n,m+1}^{a,\alpha} (x). \]

Simplifying above equations, we obtain the required recurrence relation.

**Lemma 4.2.2.** If the \( m^{\text{th}} \) order \((m \in N^0)\) moment is defined as

\[ U_{n,m}^{a,\alpha} (x) := K_{n}^{a,\alpha} (t^m, x) = n \sum_{k=0}^{\infty} b_{n,k}^{a,\alpha} (x) \left( \frac{(k+1)\alpha}{m} \right) t^m dt, \]

then we have

\[ U_{n,m}^{a,\alpha} (x) = \frac{1}{m+1} \sum_{j=0}^{m} \left( \begin{array}{c} m+1 \\ j \end{array} \right) \frac{1}{n^{m-j}} T_{n,j}^{a,\alpha} (x). \]
Consequently from Lemma 4.2.1, we obtain

\[ U_{n,0}^{\alpha,\alpha}(x) = 1, \quad U_{n,1}^{\alpha,\alpha}(x) = \frac{1}{n} \left[ \frac{1}{2} + nx + \frac{a\alpha x}{1 + \alpha x} \right] \]

and

\[ U_{n,2}^{\alpha,\alpha}(x) = \frac{1}{n^2} \left[ \frac{1}{3} + n^2 x^2 + n \left( \alpha x^2 + 2x + \frac{2a\alpha x^2}{1 + \alpha x} \right) + \frac{a^2 \alpha^2 x^2}{(1 + \alpha x)^2} + \frac{2a\alpha x}{1 + \alpha x} \right] \]

**Proof:**

\[ U_{n,m}^{\alpha,\alpha}(x) = n \sum_{k=0}^{\infty} b_{n,k}^{\alpha,\alpha}(x) \int_{k/n}^{(k+1)/n} t^m \, dt \]

\[ = \frac{n}{m+1} \sum_{k=0}^{\infty} b_{n,k}^{\alpha,\alpha}(x) \left( \frac{k+1}{n} \right)^{m+1} - \left( \frac{k}{n} \right)^{m+1} \]

\[ = \frac{n}{m+1} \sum_{k=0}^{\infty} b_{n,k}^{\alpha,\alpha}(x) \sum_{j=0}^{m+1} \binom{m+1}{j} \left( \frac{k}{n} \right)^j \left( \frac{1}{n} \right)^{m+1-j} - \left( \frac{k}{n} \right)^{m+1} \]

\[ = \frac{1}{m+1} \sum_{k=0}^{\infty} \sum_{j=0}^{m+1} \binom{m+1}{j} \frac{1}{n} \sum_{k=0}^{\infty} b_{n,k}^{\alpha,\alpha}(x) \frac{k}{n} \]

\[ = \frac{1}{m+1} \sum_{j=0}^{m+1} \binom{m+1}{j} \frac{1}{n} T_{n,j}^{\alpha,\alpha}(x). \]

Other results can be obtained by putting \( m = 0, m = 1 \) and \( m = 2 \).

**Remark 4.2.1.** By Lemma 4.2.2, we obtain

\[ K_n^{\alpha,\alpha}(t-x, x) = \mu_{n,1}^{(\alpha,\alpha)}(x) = \frac{1}{n} \left[ \frac{1}{2} + \frac{a\alpha x}{1 + \alpha x} \right]. \]

\[ K_n^{\alpha,\alpha}((t-x)^2, x) = \mu_{n,2}^{(\alpha,\alpha)}(x) = \frac{1}{3n} + \frac{1}{n} \left[ \alpha x^2 + x \right] + \frac{a^2 \alpha^2 x^2}{n^2 (1 + \alpha x)^2} + \frac{2a\alpha x}{n^2 (1 + \alpha x)}. \]

\[ K_n^{\alpha,\alpha}((t-x)^r, x) = \mu_{n,r}^{(\alpha,\alpha)}(x) = \mathcal{O}(n^{-\frac{r+1}{2}}) \forall x \in [0, \infty), \]

where \([p] = \text{integral part of } p\) .
For sufficiently large \( n \) and \( \gamma > 1, x \in (0, \infty) \), we have the following inequality

\[
K_n^{a,a}((t-x)^2, x) \leq \frac{\gamma x (1 + \alpha x)}{n}.
\]

### 4.3. DIRECT ESTIMATES

**Theorem 4.3.1.** Let \( f \) be a bounded integrable function on \([0, \infty)\) and \( f'' \) exists at a point \( x \in [0, \infty) \), then

\[
\lim_{n \to \infty} n[K_n^{a,a}(f, x) - f(x)] = \left[ \frac{1}{2} + \frac{a \alpha x}{1 + \alpha x} \right] f'(x) + \frac{1}{2} [\alpha x^2 + x] f''(x).
\]

**Proof:** By the Taylor’s expansion of \( f \), we have

\[
f(t) = f(x) + f'(x)(t-x) + \frac{1}{2} f''(x)(t-x)^2 + r(t, x)(t-x)^2,
\]

where \( \lim_{t \to x} r(t, x) = 0 \). Operating \( K_n^{a,a} \) to the identity given by Eqn. (4.4), we obtain

\[
K_n^{a,a}(f, x) - f(x) = K_n^{a,a}(t-x, x)f'(x) + K_n^{a,a}\left((t-x)^2, x\right)\frac{f''(x)}{2} + K_n^{a,a}\left(r(t, x)(t-x)^2, x\right)
\]

Using the Cauchy-Schwarz inequality, we have

\[
K_n^{a,a}\left(r(t, x)(t-x)^2, x\right) \leq \sqrt{K_n^{a,a}\left(r^2(t, x), x\right)} \sqrt{K_n^{a,a}\left((t-x)^4, x\right)}.
\]

In view of Remark 4.2.1, we have

\[
\lim_{n \to \infty} K_n^{a,a}\left(r^2(t, x), x\right) = r^2(x, x) = 0.
\]

Now from Eqn. (4.5), (4.6) and from Lemma 4.2.2, we get

\[
\lim_{n \to \infty} nK_n^{a,a}\left(r(t, x)(t-x)^2, x\right) = 0.
\]

Thus

\[
\lim_{n \to \infty} n[K_n^{a,a}(f, x) - f(x)] = \lim_{n \to \infty} n[K_n^{a,a}(t-x, x)f'(x) + \frac{1}{2} f''(x)K_n^{a,a}\left((t-x)^2, x\right) + K_n^{a,a}\left(r(t, x)(t-x)^2, x\right)].
\]

The result follows immediately by applying Remark 4.2.1. This completes the proof of asymptotic formula.
Definition 4.3.1. Let \( C_\beta[0, \infty) \) be the space of all real-valued continuous and bounded function on \([0, \infty)\) with the norm \( \|f\| = \sup_{x \in [0, \infty)} |f(x)| \).

The modulus of continuity of the function \( f \in C_\beta[0, \infty) \) is defined by
\[
\omega(f, \delta) := \sup \|f(x) - f(y)\| : x, y \in [0, \infty), |x - y| \leq \delta.
\]

For \( f \in C_\beta[0, \infty) \) and \( \delta > 0 \), the modulus of continuity of second order is defined as
\[
\omega_2(f, \delta) := \sup \|f(x + h) - 2f(x) + f(x - h)\| : x, x \pm h \in [0, \infty), 0 \leq h \leq \delta.
\]

Definition 4.3.2. Let
\[
K_2(f, \delta) = \inf_{g \in C_\beta^2[0, \infty)} \{\|f - g\| + \delta \|g''\| : g \in C_\beta^2[0, \infty)\},
\]
be the Peetre’s \( K \)-functional, and
\[
C_\beta^2[0, \infty) = \{g \in C_\beta[0, \infty) : g', g'' \in C_\beta[0, \infty)\}.
\]

For \( f \in C_\beta[0, \infty) \), denote
\[
f_h(x) = \frac{4}{h^2} \int_0^h \int_0^h [2f(x + u + v) - f(x + 2(u + v))] du \, dv,
\] \hspace{1cm} (4.7)
the Steklov mean. The following inequalities hold:
\begin{enumerate}
  \item \( \|f_h - f\| \leq \omega_2(f, h) \).
  \item \( f'_h, f''_h \in C_\beta[0, \infty) \) and \( \|f'_h\| \leq \frac{5}{h} \omega(f, h) \), \( \|f''_h\| \leq \frac{9}{h^2} \omega_2(f, h) \).
\end{enumerate}

Theorem 4.3.2. Let \( f \in C_\beta[0, \infty) \), then for every \( x \geq 0 \), the following inequality holds
\[
|K^{a,\alpha}_n(f, x) - f(x)| \leq 5\omega\left(f, \sqrt{\mu^{(a,\alpha)}_{n,2}}(x)\right) + \frac{13}{2} \omega_2\left(f, \sqrt{\mu^{(a,\alpha)}_{n,2}}(x)\right),
\]
where \( \mu^{(a,\alpha)}_{n,2}(x) \) is as mentioned in Remark 4.2.1.

Proof: Using the Steklov mean \( f_h \) that is given by Eqn. (4.7), we obtain
\[
|K^{a,\alpha}_n(f, x) - f(x)| \leq K^{a,\alpha}_n\left(|f - f_h|; x\right) + |K^{a,\alpha}_n\left(f_h - f\right)(x); x\right) + |f_h(x) - f(x)|. \hspace{1cm} (4.8)
\]
But

\[ |K_n^{a,a}(f, x)| \leq n \sum_{k=0}^{n} |h_n^{a,a}(x)| \|f(t)\| dt \leq \|f\|. \tag{4.9} \]

Using property \( (i) \) of Steklov mean and Eqn. (4.9), we get

\[ K_n^{a,a}(f - f_h; x) \leq \|f - f_h\| \leq \omega_2(f, h). \]

By Taylor’s expansion and Cauchy-Schwarz inequality, we have

\[ \left| K_n^{a,a}(f_h - f_h; x) \right| \leq \|f_h\| \sqrt{\frac{1}{2} \|f_h\|^2} \right| \left| K_n^{a,a}(t-x)^2; x \right|. \]

Applying property \( (ii) \) of Steklov mean, for sufficiently large \( n \), we obtain

\[ \left| K_n^{a,a}(f_h - f_h; x) \right| \leq \frac{5}{h} \omega(f, h) \sqrt{\mu_{n,2}^{(a,a)}(x)} + \frac{9}{2h^2} \omega_2(f, h) \mu_{n,2}^{(a,a)}(x). \]

Choosing \( h = \sqrt{\mu_{n,2}^{(a,a)}(x)} \), and substituting the values of the above estimates in Eqn. (4.8), we get the desired relation.

**Theorem 4.3.3.** Let \( f \in C^1_b[0, \infty) \), then for every \( x \geq 0 \), the following inequality holds

\[ \left| K_n^{a,a}(f, x) - f(x) \right| \leq \left| \mu_{n,1}^{(a,a)}(x) \right| |f'(x)| + 2 \sqrt{\mu_{n,2}^{(a,a)}(x)} \omega(f', \mu_{n,2}^{(a,a)}(x)). \]

\( \mu_{n,1}^{(a,a)}(x) \) and \( \mu_{n,2}^{(a,a)}(x) \) are central moments of first and second order.

**Proof:** For any \( t \in [0, \infty), x \in [0, \infty) \), we have

\[ f(t) - f(x) = f'(x)(t-x) + \int_x^t (f'(u) - f'(x))du. \]

Applying \( K_n^{a,a}(x,x) \) to both sides of the above relation, we get

\[ K_n^{a,a}(f(t) - f(x), x) = f'(x)K_n^{a,a}(t-x, x) + K_n^{a,a} \left( \int_x^t (f'(u) - f'(x))du, x \right). \]

Using the well known property of the modulus of continuity

\[ |f(u) - f(x)| \leq \omega(f, \delta) \left( \frac{|u-x|}{\delta} + 1 \right), \delta > 0, \]

we obtain

\[ \left| \int_x^t (f'(u) - f'(x))du \right| \leq \omega(f', \delta) \left( \frac{(t-x)^2}{\delta} + |t-x| \right). \]
Therefore,
\[ |K_n^{\alpha,\alpha}(f,x) - f(x)| \leq |f'(x)| \cdot |K_n^{\alpha,\alpha}(t-x,x)| + \omega(f',\delta) \left( \frac{1}{\delta} K_n^{\alpha,\alpha}((t-x)^2,x) + K_n^{\alpha,\alpha}(|t-x|,x) \right). \]

Applying Cauchy-Schwarz inequality, we get
\[ |K_n^{\alpha,\alpha}(f,x) - f(x)| \leq |f'(x)| \cdot |K_n^{\alpha,\alpha}(t-x,x)| + \omega(f',\delta) \left( \frac{1}{\delta} \sqrt{K_n^{\alpha,\alpha}((t-x)^2,x)} + 1 \right) \sqrt{K_n^{\alpha,\alpha}((t-x)^2,x)}. \]

Choosing \( \delta = \sqrt{\mu_{n,2}^{(\alpha,\alpha)}(x)} \), the required result follows.

4.4. CONVERGENCE ESTIMATE

Now we shall estimate the convergence estimate for the operator defined by Eqn. (4.3) for the function with derivatives of bounded variation for all \( x \in (0,\infty) \) where left hand limit and right hand limit of \( f'(x) \) exist. It has been an interesting area in the theory of approximation after the work of Bojanic and Cheng (1989) who estimated the rate of convergence with the derivative of bounded variation for Bernstein and Hermite-Fejer polynomials. After that many researchers have obtained such type of results for different linear positive operators. In this direction Gupta et al. (2005) has obtained such type of estimates for the beta operator of the second type by decomposition techniques.

Let us define the class \( \sigma \geq 0 \) of all functions, having a derivative of bounded variation on every finite sub interval of \( [0,\infty) \) with polynomial growth \( |f(x)| \leq Cx^\sigma \), \( \forall x > 0 \).

Remark 4.4.1. If \( g(y) \) is a function of bounded variation on each finite sub interval of \( (0,\infty) \) then \( f \in BV_{\sigma}^D \) can be represented by following integral modification
\[ f(x) = \int_0^y g(y)dy + f(0), \]

Similar behavior is shown by the operator defined by Eqn. (4.3) as it can also be written as
\[ K_n^{\alpha,\alpha}(f,x) = \int_0^\infty W_n^{\alpha,\alpha}(x,y)f(y)dy, \]

where
\[ W_n^{\alpha,\alpha}(x,y) = n \sum_{k=0}^\infty j_n^{\alpha,\alpha}(x)\xi_{n,k}(y). \]

\( \xi_{n,k}(y) \) is the characteristic function of the interval \( \left[ \frac{k}{n}, \frac{k+1}{n} \right] \) with respect to \([0,\infty)\).
Lemma 4.4.1. For certain \( x \in [0, \infty) \), \( \gamma > 1 \) and sufficiently large \( n \), we have

1. \[ \int_0^x W_n^{a,\alpha}(x, y) dy = \beta_n^{a,\alpha}(x, u) \leq \frac{1}{(x-u)^2} \frac{\gamma(1+\alpha\alpha)}{n}, 0 \leq u < x, \]

2. \[ \int_0^x W_n^{a,\alpha}(x, y) dy = 1 - \beta_n^{a,\alpha}(x, v) \leq \frac{1}{(v-x)^2} \frac{\gamma(1+\alpha\alpha)}{n}, x \leq v < \infty. \]

Proof: For \( y \in [0, u] \), since \( 0 \leq u < x \), we have

\[ \frac{x-y}{x-u} \geq 1 \]

Now by Lemma 4.2.2 and Remark 4.2.1, we obtain the first result by simple computation and similarly second result has been obtained.

Theorem 4.4.1. Let \( f \in BV_{\sigma}^D \) then for sufficiently large \( n \) and for every \( x \in [0, \infty) \), we have

\[ |K_n^{a,\alpha}(f, x) - f(x)| \leq \frac{1}{n} \left| \frac{1}{2} + \frac{a\alpha\alpha}{1+\alpha\alpha} \left| \frac{f'(x+) + f'(x-)}{2} + \sqrt{\frac{\gamma(1+\alpha\alpha)}{n}} \left| f'(x+) + f'(x-) \right| \right| \right| \]

\[ + \frac{\gamma(1+\alpha\alpha)}{n} \sum_{i=1}^{\frac{\sqrt{n}}{x}} \sqrt{f'(x)_i} + x^n^{-1/2} \sqrt{x} \sum_{x^{-x}}^{x^n} \sqrt{f'(x)} \]

\[ J_{1.6} \leq \frac{\gamma(1+\alpha\alpha)}{n} \sum_{i=1}^{\frac{\sqrt{n}}{x}} \sqrt{f'(x)_i} + x^n^{-1/2} \sqrt{x} \sum_{x^{-x}}^{x^n} \sqrt{f'(x)}. \]

where \( \sqrt{\cdot} f'(x) \) denote the total variation of \( f'(x) \) on \([a, b]\) and an auxiliary function \( f'_x \) is defined by

\[ f'_x(y) = \begin{cases} f'(y) - f'(x_-), & 0 \leq y < x \\ 0, & y = x \\ f'(y) - f'(x_+), & x < y < \infty. \end{cases} \]

Proof: From Remark 4.4.1, for \( r \in [0, \infty) \) we obtain

\[ K_n^{a,\alpha}(f, x) - f(x) = \int_0^x W_n^{a,\alpha}(x, y)(f(y) - f(x)) dy \]

\[ = \int_0^x W_n^{a,\alpha}(x, y)\int_x^y f'(r) dr dy. \]
Now we may write
\[
f'(r) = f'_x(r) + 1/2(f'(x+) + f'(x-)) + 1/2(f'(x+) - f'(x-))\text{sgn}(r-x),
\]
\[+ \Psi_x'(r)[f'(r) - 1/2(f'(x+) + f'(x-))]
\]
where \(\Psi_x'(r)\) is a characteristic function defined as
\[
\Psi_x'(r) = \begin{cases} 
1 & r = x \\
0 & r \neq x.
\end{cases}
\]

Therefore, we can write
\[
K_n^{a,a}(f, x) - f(x) = \int_0^x \left(\int_y^x f'_r(r)dr\right)W_n^{a,a}(x, y)dy
\]
\[+ \int_0^y \left(\int_y^x 1/2(f'(x+) + f'(x-))dr\right)W_n^{a,a}(x, y)dy
\]
\[+ \int_0^y \left(\int_y^x 1/2(f'(x+) - f'(x-))\text{sgn}(r-x)dr\right)W_n^{a,a}(x, y)dy
\]
\[+ \int_0^y \left(\int_y^x \Psi_x'(r)(f'(t) - 1/2(f'(x+) + f'(x-)))dr\right)W_n^{a,a}(x, y)dy
\]
\[= J_1 + J_2 + J_3 + J_4 \quad \text{(say)}.
\]

Taking modulus both sides, we obtain
\[
|K_n^{a,a}(f, x) - f(x)| \leq |J_1| + |J_2| + |J_3| + |J_4|.
\]

Now we have to estimate \(J_1, J_2, J_3\) and \(J_4\)
\[
|J_1| = \left|\int_0^x \left(\int_y^x f'_r(r)dr\right)W_n^{a,a}(x, y)dy\right| \quad (4.11)
\]
\[
= \left|\int_0^y \left(\int_y^x f'_r(r)dr\right)W_n^{a,a}(x, y)dy + \int_y^x \left(\int_y^x f'_r(r)dr\right)W_n^{a,a}(x, y)dy\right|
\]
\[
\leq \left|\int_0^y \left(\int_y^x f'_r(r)dr\right)W_n^{a,a}(x, y)dy\right| + \left|\int_y^x \left(\int_y^x f'_r(r)dr\right)W_n^{a,a}(x, y)dy\right|
\]
\[= J_{1.5} + J_{1.6} \quad \text{(Say)}.
\]

Estimation of \(J_{1.5}\) and \(J_{1.6}\) is required now. Using Integration by part and Lemma 4.4.1 we have
\[
J_{1.5} = \left|\int_0^y \left(\int_y^x f'_r(r)dr\right)d_y \beta_n^{a,a}(x, y)dy\right|
\]
\[
= \left|\int_0^x \beta_n^{a,a}(x, y)f'_r(y)dy\right|.
\]
Decomposing \([0, x]\) into two parts taking \(t = x - x/\sqrt{n}\) and using Lemma 4.2.3 and the inequality \(\int_{\mathbb{R}} d_s \beta^{\alpha, \alpha}_n (x, y) \leq 1\forall [a, b] \subseteq [0, \infty),\) we have

\[
J_{1.5} \leq \left| \int_0^x \beta^{\alpha, \alpha}_n (x, y) f'_s(y) dy \right| + \left| \int_0^x \beta^{\alpha, \alpha}_n (x, y) f'_s(y) dy \right|
\]

\[
\leq \int_0^x \left| \beta^{\alpha, \alpha}_n (x, y) \right| f''_s(y) dy + \int_0^x \left| \beta^{\alpha, \alpha}_n (x, y) \right| f'_s(y) dy
\]

\[
\leq \frac{\gamma(1 + \alpha\gamma)}{n} \int_0^x \sqrt{\frac{x}{(x-y)^2}} \frac{1}{x} dy + \frac{x}{\sqrt{n}} \int_{x-(x/\sqrt{n})}^x f'_s.
\]

Now by taking substitution \(v = \frac{x}{x-y},\) we obtain

\[
\frac{\gamma(1 + \alpha\gamma)}{n} \int_0^x \sqrt{\frac{x}{(x-y)^2}} \frac{1}{x} dy = \frac{\gamma(1 + \alpha\gamma)}{n} \int_{\sqrt{x/(x-v)}}^x f'_s dv
\]

\[
\leq \frac{\gamma(1 + \alpha\gamma)}{n} \sum_{i=1}^{\sqrt{n}} \sqrt{\frac{x}{x-(x/i)}} f'_s dv
\]

\[
\leq \frac{\gamma(1 + \alpha\gamma)}{n} \sum_{i=1}^{\sqrt{n}} \sqrt{f'_s}.
\]

Thus, we obtain the result

\[
J_{1.5} \leq \frac{\gamma(1 + \alpha\gamma)}{n} \sum_{i=1}^{\sqrt{n}} \sqrt{f'_s} + \frac{x}{\sqrt{n}} \sqrt{f'_s}.
\]  \hspace{1cm} (4.12)

Similarly, by using Lemma 4.4.1 and \(w = x + x/\sqrt{n}\), we obtain

\[
J_{1.6} \leq \frac{\gamma(1 + \alpha\gamma)}{n} \sum_{i=1}^{\sqrt{n}} \sqrt{f'_s} + \frac{x}{\sqrt{n}} \sqrt{f'_s}.
\]  \hspace{1cm} (4.13)

Now the second estimate is given by Lemma 4.2.2 and Remark 4.2.1 as follows

\[
J_2 = 1/2 (f'(x+) + f'(x-)) \int_{\mathbb{R}} (y-x) W^{\alpha, \alpha}_n (x, y) dy
\]

\[
\left| J_2 \right| = \frac{1}{2n} \left| f'(x+) + f'(x-) \right| \frac{1 + \alpha\gamma}{1 + \alpha\gamma}.
\]  \hspace{1cm} (4.14)

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The third estimate $J_3$ is obtained by Schwarz inequality and Lemma 4.2.3 and Remark 4.2.1, we obtain

$$|J_3| \leq 1/2(f'(x+) - f''(x)) \int_0^\infty |y-x| W_n^{a,a}(x,y)dy$$

(4.15)

$$\leq 1/2 |f'(x+)-f''(x)| K_n^{a,a}(|y-x|,y)$$

$$\leq 1/2 |f'(x+)-f''(x)| (K_n^{a,a}(|y-x|^2,y))^{1/2}$$

$$\leq 1/2 |f'(x+)-f''(x)| \sqrt{\frac{\alpha(1+\alpha)}{n}},$$

and

$$\int_0^\infty (\int_y^\infty \{f'(t) - 1/2(f'(x+)+f'(x-))\}) \Psi_4(r)dr W_n^{a,a}(x,y)dy = 0$$

$$\Rightarrow |J_4| = 0.$$

Combining the estimates of $J_1$ to $J_4$, we obtain the required result. This completes the proof of the theorem.

4.5. CONCLUSION

We studied a Kantorovich type integral modification of a sequence of positive linear operators due to Gupta and Singh (2016). We obtained asymptotic expansion for the operator and error estimates using second order of the modulus of continuity, K-functional and Steklov mean. Further, we found convergence estimate for the function with derivative of bounded variation.