CHAPTER 1

INTRODUCTION

1.1. HISTORICAL BACKGROUND

Approximation theory concerning convergence of linear positive operators, deals with the study of several problems. One of the main problems is to approximate a complicated function by some simpler function. One has to minimize the error introduced thereby.

Let a set of functions \( \Phi_n(x) \) defined on a linear space \( S \) of continuous real valued function and \( f(x) \) be a function on \( S \). A linear combination of \( \Phi_n \) can be defined as

\[
P = c_1\Phi_1 + c_2\Phi_2 + c_3\Phi_3 + \ldots + c_n\Phi_n, \tag{1.1}
\]

which is close to \( f \), where \( c_i, i = 1, 2, \ldots n \) are real constants.

Thus an element \( f \) on \( S \) is called approximable by linear combination defined by Eqn. (1.1), if for each \( \epsilon > 0 \),

\[
\|f - P\| < \epsilon.
\]

Also, if the quantity

\[
E_n(f) = E_n = \inf_{c_1, \ldots, c_n} \|f - P\|, \tag{1.2}
\]

attains its infimum for some \( P \), then \( P \) is called a linear combination of best approximation of degree \( n \). The quantity \( E_n(f) \) is called error function.

The theory got its origin by the work of Chebyshev (1854-1859) on the best approximation of functions by polynomial uniformly and developed from the fundamental result given by Weierstrass (1885) known as a Weierstrass approximation theorem. The theorem states:

For any function \( f(x) \in C[a, b] \), the class of continuous, real valued and differentiable function on a closed interval \([a, b]\), there exist an algebraic polynomial \( p(x) \) for which

\[
|f(x) - p(x)| < \epsilon, \forall x \in [a, b] \text{ and } \epsilon > 0
\]
This implies that the objective is to minimize the upper limit value of \(|f(x) - p(x)|\), which is also known as error function.

Borel (1905) proposed interpolation process to find a polynomial \(P(x)\), which converges to function \(f \in C[a,b]\) uniformly. Following Borel, Bernstein (1912) was able to construct a very sound polynomial known as Bernstein polynomial and is given as

\[
B_n(f, x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f \left( \frac{k}{n} \right),
\]

for any \(f \in C[0,1]\), \(x \in [0,1]\) and \(n \in \mathbb{N}\).

The Bernstein operators provide a simple and constructive proof of Weierstrass approximation theorem for the case \(C[0,1]\). The result can be extended to \(C[a,b]\) by defining the function

\[
y : [0,1] \to [a,b], \quad y(x) = (1-x)a + xb.
\]

Sharp as well as simple approximation properties of Bernstein polynomials made it popular and most studied positive linear operators. Several important generalizations and modification were suggested by researchers like Kantorovich (1930), Schurer (1962), Durrmeyer (1967), Stancu (1968), Lupas (1987) and many more are there.

Bohman (1952) and Korovkin (1953) provide a very efficient criterion to prove uniform convergence of a sequence of positive linear operators towards the continuous function, known as Bohman-Korovkin theorem. The theorem states:

For any \(n \in \mathbb{N}\), let \(S_n : C[a,b] \to C[a,b]\) be a sequence of positive linear operators. If for \(e_n = t^n\) the following three conditions hold

\begin{align*}
a. \quad S_n(e_0, x) &= 1 + u_n(x), \\
b. \quad S_n(e_1, x) &= x + v_n(x), \\
c. \quad S_n(e_2, x) &= x^2 + w_n(x),
\end{align*}

such that \(\lim_{n \to \infty} u_n(x) = 0\), \(\lim_{n \to \infty} v_n(x) = 0\), and \(\lim_{n \to \infty} w_n(x) = 0\) uniformly on \([a,b]\), then for any \(f \in C[a,b]\) and \(x \in [a,b]\), \(\lim_{n \to \infty} S_n(f, x) = f(x)\) uniformly on \([a,b]\). Here \(e_0, e_1\) and \(e_2\) are called Korovkin’s test functions.

Practical aspect of approximation theory concerned not only with the fact of uniform convergence of a sequence, but also with the speed of convergence, that is how fast it tends to the approximating function.
Obtaining a sequence of operators which approximate a given function next desired step is to find an error between function and approximate polynomial. Degree of approximation considers the behavior of error function $E_n(f)$ defined by Eqn. (1.2) and find whether $E_n(f)$ tends to zero or not as $n \to \infty$.

The error estimates could be obtained with respect to some functional norm, seminorm, and pointwise local or global. Global result involves the whole domain of the functions, while local result concerns with the finite subsets of the domain.

If $E_n(f)$ tends to zero as $n \to \infty$ then the next question of interest is how fast it tends to zero. It is called the rate of convergence, which is an estimate of the speed at which a sequence of positive linear operator approaches its limit. This estimate can be obtained precisely by comparing $E_n(f)$ with certain standard sequences like $n^k$.

Concerning Bernstein polynomial, Voronowskaja (1932) obtained the following estimate:

Let $f(x)$ is at least twice differentiable at a fixed point $x \in [0,1]$ then

$$B_n(x) - f(x) = \frac{-x(1-x)}{2n} f''(x) + O(n^{-1})$$

This type of result is usually called as Voronowskaja kind asymptotic formula.

Approximation order is directly proportional to the smoothness properties of functions. Smoothness of the function is given by its modulus of continuity therefore more precise estimate can be obtained by using the modulus of continuity, which is defined as

$$\omega(f, \delta) := \sup \{ |f(x) - f(y)| : x, y \in [a, b], |x - y| \leq \delta \}, \delta \geq 0.$$  

Popoviciu (1935) found error estimate for Bernstein polynomial in terms of modulus of continuity, which states

$$\max_{x \in [0,1]} |B_n(f, x) - f(x)| \leq \frac{5}{4} \omega(f, n^{-1/2}).$$

The quantitative result to find approximation order using the modulus of continuity was given by Shisha and Mond (1968) in following relation:

$$|S_n(f; x) - f(x)| \leq |f(x)||S_n(1; x) - 1| + (S_n(1; x) + 1)\omega(f, S_n((t - x)^2; x^{1/2}),$$

where $S_n$ is a sequence of linear positive operators.
Committed error can be estimated by another important tool called the modulus of continuity of second order for which the first general quantitative result was given by Esser (1976) and later enhanced by Gonska (1984). These results of a combination of the first and second order of the modulus of continuity give better estimates as they include decomposition of error function into deviation from Korovkin’s test functions. A much refined result was established by Păltănea (1995) stated as follows:

\[
|S_n(f;x) - f(x)| \leq |f(x)| \cdot |S_n(e_0;x) - 1| + 1/\delta |S_n(e_1 - x;x)| \cdot \omega_1(f,\delta) \\
+ \left[ S_n(e_o;x) + \frac{1}{2\delta^2} S_n((e_1 - x)^2;x) \right] \cdot \omega_2(f,\delta),
\]

where \( I = [a,b] \), \( I_1 \subset I \), \( S_n: C(I) \to C(I_1) \) and \( S_n \) is a sequence of linear positive operators.

For \( 0 < \delta \leq \frac{1}{2} (b - a) \), the \( m \)-th order modulus of continuity of a function continuous on \( [a, b] \) is defined as

\[
\omega_n(f,\delta,[a,b]) = \sup \left\{ \Delta^m_h f(x) \mid h \leq \delta;x,x+h \in [a,b] \right\},
\]

\( \Delta^m_h f(x) \) is the \( m \)-th order forward difference of \( f(x) \) with step length \( h \).

Most of the known linear positive operators reproduce linear functions and so this property should also be contained in point wise estimate of the operator in concern. Such an accomplishment could not be attained by estimates given in terms of first order modulus of continuity denoted by \( \omega_1 \). It was fulfilled by second order modulus of continuity denoted by \( \omega_2 \) as linear functions can be desolated by it. Therefore, it is advantageous to measure the order of approximation by the second order of the modulus of continuity.

The theory of approximation is very rich with varied studies done for different problems and in different directions by generations of mathematicians from all over the world. We include some specific problems of approximation of a function by linear positive operator and noted the contribution made by eminent mathematicians.

An important development in the theory of approximation was integral modification of summation type operators (Bernstein polynomial), which is due to Kantorovich (1930). It was observed that these operators were not \( L_p \)-approximation method. Kantorovich introduced integral modification of the Bernstein polynomials to consider larger class \( L_1[0,1] \) of functions Lebesgue integrable on \([0,1]\).

\[
K_n(f,x) = (n + 1) \sum_{k=0}^{n} \left( \int_{t_k}^{t_{k+1}} f(t) dt \right) p_{n,k}(x),
\]
where \( p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k} \) and \( I_K = \left[ \frac{k}{n+1}, \frac{k+1}{n+1} \right] \).

These operators are called summation-integral type which is also known as linear smoothing operator. Durrmeyer (1967) introduced another modification of Bernstein polynomials to approximate Lebesgue integrable functions \( f \) on \([0,1]\) as

\[
D_n(f,x) = (n + 1) \sum_{k=0}^{n} \left( \int_{0}^{1} p_{n,k}(x)f(t)dt \right) p_{n,k}(x),
\]

where \( p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k} \).

Approximation properties of these operators were broadly studied by Derriennic (1981). Inspired by the work of Durrmeyer and Derriennic, integral modification of Lupaş operators was introduced by Sahai and Prasad (1985). Umar and Razi (1985) extended the studies for Kantorovich-type integral modification of the operators, due to Jain (1972). Integral modification of Szász-Mirakyan operators were introduced by Mazhar and Totik (1985) and by Kasana et al. (1985) in the same sense to approximate functions, Lebesgue integrable on \([0,\infty)\). Very recently Sidharth et al. (2017) proposed Szász-Durremeyer type operators based on Boas-Buck type polynomials which includes Brenke, Sheffer and Appell polynomials as particular cases. They estimated some direct estimates. Also Aral et al. (2017) constructed another new sequence of Szász-Mirakyan type operators and studied some approximation problems. Gal et al. (2016) studied some approximation properties of Szász type operators in compact disks.

In continuation, several new sequences on summation and integral type have been introduced and studied by numerous researchers. Some important integral operators are the Durrmeyer variant of Baskakov-Szász operators due to Gupta and Srivastava (1993), Kantorovich type modification of Bleimann, Butzer and Hahn operators due to Abel and Ivan (2002). Durrmeyer type variant of mixed Meyer–König-Zeller operators (MKZ) operators due to Abel et al. (2003), Bézier variant of MKZ operators due to Gupta (2003) considered by Gupta and Abel (2004). A general sequence of operators was introduced by Srivastava and Gupta (2003) which includes some classical operators as special cases like Phillips operators, Baskakov-Durrmeyer operators and Bernstein-Durrmeyer operators. To approximate integrable functions on \([0,\infty)\), Gupta and Lupaş (2003) considered by Gupta and Abel (2004). A general sequence of operators was introduced by Srivastava and Gupta (2003) which includes some classical operators as special cases like Phillips operators, Baskakov-Durrmeyer operators and Bernstein-Durrmeyer operators. To approximate integrable functions on \([0,\infty)\), Gupta and Lupaş (2005) introduced a mixed sequence of operators by modifying Szász operators with the weights of Beta basis functions. Integral modification of Jain operators with Beta basis function was introduced by Tarabie (2013). Agrawal and Goyal (2015) proposed a generalized Kantorovich variant of Baskakov operators.
The core problems of approximation by positive linear operators include the study of the direct theorem, inverse theorem, rate of convergence and asymptotic behavior. These results make important theoretical and practical aspects of study of linear positive operators.

The results, which infer the degree of approximation from given characteristics of functions, are called direct results. The initial result was given by Jackson (1930). Direct theorems enable one to estimate from Eqn. (1.2), the error in an approximation of a function belongs to certain classical approximation classes like class of algebraic and trigonometric polynomial, spline etc. It gives an upper bound on the rate of approximation. Mathematically, it can be explained as follows:

Let Y subset of X is a class of functions sharing certain characteristics with particular f (e.g. \( \| f^{\alpha} \| \leq 1 \)). The results of the type

\[ f \in Y \Rightarrow E_n(f) = O(n^{-\alpha}) \]

are known to be as Jackson type (or direct) results. For instance, if \( f(x) \) belongs to the Lipschitz class ‘Lip \( \alpha \)’ or in other word

If \( (\delta) \leq M \delta^\alpha, 0 < \alpha < 1 \), then \( |f(x) - P_n(x)| = O(n^{-\alpha}) \).

Corresponding to direct result a converse result also there in existence, which determine structural characteristics of functions from their order of approximation, known as inverse theorems. The Inverse theorem provides lower bound on approximation error. The theorem was first given by S. N. Bernstein (1912). Mathematically, it can be explained as follows:

For set Y, we have

\[ E_n(f) = O(n^{-\alpha}) \Rightarrow f \in Y. \]

For instance:

If \( |f(x) - P_n(x)| = O(n^{-\alpha}) \) then \( f(x) \) belongs to the Lipschitz class ‘Lip \( \alpha \)’ or in other word if \( \omega(\delta) \leq M \delta^\alpha, 0 < \alpha < 1 \).

Above results do not contain the case of \( \alpha = 1 \). The estimate for \( \alpha = 1 \) was finally established by Zygmund (1945) involving second differences rather than first differences (see section 1.2.7). Many mathematicians extended the problem to more generalized approximation process. The textbooks of Timan (1963) and of DeVore and Lorentz (1993) contain very rich information on direct and inverse theorems for approximation by algebraic and trigonometric polynomials.
One of the important results in approximation theory is the saturation theorem. The theorem shows that the order of approximation estimated in the direct theorem cannot be improved further by assuming higher smoothness of the function. Saturation theorem was first developed formally by Favard (1949), can be stated as follows:

The sequence of positive linear operator \( \langle S_n \rangle: C[a, b] \to C[c, d] \) is saturated on \([c,d] \subset [a, b] \), if there is a sequence of positive functions \( \langle \Psi_n \rangle \) on \([c, d] \) which tends to zero, as \( n \) tends to \( \infty \), such that \( \|g - S_n(g)\| = o(\Psi_n) \) if and only if \( g \) is constant and there is a non constant function \( g_0 \) in \( C[a, b] \) such that

\[
\|g_0 - S_n(g)\| = O(\Psi_n).
\]

\( T(S_n) \) is called trivial class for which first condition holds. Thus, the order of approximation above a certain limit \( O(\Psi_n) \), \( \Psi_n \to 0 \) as \( n \to \infty \) is possible only for a trivial class. The sequence \( \langle \Psi_n \rangle \) is called the saturation order of linear positive operator \( \langle S_n \rangle \) and the set of those functions in \( C[a, b] \) satisfying second condition is called saturation class of \( \langle S_n \rangle \).

Saturation theorem due to Favard has been found to be very useful by many mathematicians to obtain an optimal rate of convergence that can be obtained from the nontrivial class of functions.

Direct, inverse and saturation theorem have been extensively studied by several researchers. Saturation result in the algebraic case was also originated by Leew (1959), Bajanski and Bojanic (1964) and Lorentz (1964). They obtained the results for Bernstein polynomials. Sunouchi (1965 and 1969), Suzuki (1965 and 1967), Butzer and Berens (1967), Shapiro (1969) and Berens and Lorentz (1972) obtained the direct, inverse and saturation results by semi-groups of operators and in local approximation. May (1976) obtained direct, inverse and saturation theorems for exponential type operators which includes Bernstein operators, Szasz-Mirakyan operators and Baskakov operators as special cases. Some other important contributions in this direction are due to Ditizian and May (1976a, 1976b). Liu (1986) established a converse theorem for Bernstein–Bézier polynomials. Heilmann (1988) established saturation theorem for modified Bernstein operators in \( L_p \) approximation. Kasana et al. (1991) estimated the direct, inverse, and saturation theorems for linear combination of Durrmeyer operators. They obtain saturation and inverse results for modified Bernstein polynomials. Liu (1991) also studied Bernstein–Kantorovich–Bézier operators in the \( L_p \)-norm and estimated the direct result. May (1997) considered combinations due to Butzer (1953) to improve the order of approximation of the Phillips operators and established some direct, inverse, and saturation results. Felten (1998) obtained direct and inverse theorem for Bernstein-type operators using Ditzian Totik modulus of second order. Finta (2005) estimated converse theorem of integral modification of Baskakov operator in terms of unified K-functional. Qi and Zhang (2009) obtained the point wise approximation and equivalence theorems using the modulus of smoothness. In 2011, Feng (2011)
introduced a new norm and a new K-functional and established direct and converse theorems for the Baskakov operators with the Jacobi-type weight function. Gupta and Rassias (2015) found direct estimates for Durrmeyer variant of modified Szász type operators consisting basis functions due to Jain (1972) and Păltănea (2008).

It was observed that a sequence of linear positive operators can be constructed easily, but from Korovkin’s theorem its optimal rate of convergence could not be better than $O(n^{-2})$, however smooth the function may be. The order of approximation can be improved by reducing positivity condition. A technique to attain this objective is linear combinations which were first considered by Butzer (1953) to improve the approximation order of Bernstein polynomials. He considered the following linear combinations of Bernstein polynomials.

$$B_n(f,k,x) = \sum_{j=0}^{k} C(j,k) B_{2j,n}(f,x),$$

where

$$C(j,k) = \prod_{i=0}^{k} \left( \frac{2^j}{2^j - 2^i} \right), \quad k \neq 0 \text{ and } C(0,0) = 1.$$  

Rathore (1973) and May (1976) considered more general linear combinations to improve the order of approximation of exponential type operators. Let $a_0, a_1, a_2, \ldots, a_k$ be $(k+1)$ arbitrary but fixed distinct positive integers. Then the operator $L_n(f,k,x)$ is called the linear combinations of $L_{a,j,n}(f,x)$ if

$$L_n(f,k,x) = \frac{1}{\Delta} \begin{vmatrix} L_{a_0,n} & a_0^{-1} & a_0^{-2} & \ldots & a_0^{-k} \\ L_{a_1,n} & a_1^{-1} & a_1^{-2} & \ldots & a_1^{-k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ L_{a_k,n} & a_k^{-1} & a_k^{-2} & \ldots & a_k^{-k} \end{vmatrix}$$

with

$$\Delta = \begin{vmatrix} 1 & a_0^{-1} & a_0^{-2} & \ldots & a_0^{-k} \\ 1 & a_1^{-1} & a_1^{-2} & \ldots & a_1^{-k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_k^{-1} & a_k^{-2} & \ldots & a_k^{-k} \end{vmatrix}$$

After simplify, we can write above expression as

$$L_n(f,k,x) = \sum_{i=0}^{k} C(i,k) L_{a,i,n}(f,x),$$
where
\[
C(i, k) = \prod_{\substack{j=0 \\
i \neq j}}^{k} \left( \frac{a_i}{a_i - a_j} \right), \quad k \neq 0 \text{ and } C(0, 0) = 1.
\]


Another technique called iterative combination was introduced by Micchelli (1969) to improve the order of the approximation of Bernstein polynomial. The \( r \)th iterative combinations of the Bernstein polynomial are defined as
\[
T_{n,r}(f, x) = (I - (1 - B_n)^r)(f(t), x) = \sum_{i=1}^{r} (-1)^{i+1} \binom{r}{i} B_{n}^{i}(f, x),
\]
where \( B_{n}^{i}(f, x) \) denotes the \( i \)th iteration and \( B_{n}^{0}(f, x) = I \).

Later Agarwal and Kasana (1984) gave a sharp estimate of the result due to Micchelli (1969) and established the result for asymptotic approximation of sufficiently smooth functions for Bernstein polynomials. Gupta and Vasishtha (2004) considered the iterative combinations of Beta-Szász type operators (see Gupta (2003)) and established direct theorems. In the same sense Agrawal et al. (2007) obtained estimates of

Another problem of interest in approximation theory is for derivatives i.e. simultaneous approximation. Approximation of the derivative of a function by the corresponding order derivative of the operators is called simultaneous approximation. It was initiated by the work of Lorentz (1953), who showed:

If a function $f$ is bounded in $[0, 1]$ and its $r$th derivative exists at $x_0 \in [0,1]$, then

$$P_n^{(r)}(f, x_0) \to f^{(r)}(x_0)$$


Mihesen (1998), which includes well-known Baskakov–Durrmeyer and Szász–Durrmeyer operators as special cases. They studied the convergence estimates in simultaneous approximation.

An approximation of a function of bounded variation is one more field of interest in theory. Functions of bounded variation of a single variable were earliest introduced by Jordan (1881) dealing with the convergence of Fourier series. A function is supposed to be of bounded variation if and only if it can be represented as the difference of two increasing or decreasing function.

The classical result due to Jordan (1981) states that if \( f \) is of bounded variation on \([0,1]\), then for each \( x \) in \((0,1)\),

\[
\lim_{n \to \infty} B_n(f, x) = \frac{1}{2} \left( f(x + 0) + f(x - 0) \right),
\]

where \( B_n \) is the Bernstein polynomial.

weight function of Beta basis function and established rate of convergence for functions having derivatives of bounded variation.

A few decades ago a general class of linear positive operator has been introduced by Stancu (1983). Based on non-negative parameters $\alpha$ and $\beta$, Stancu (1983) considered the following form of operators

$$S_m^{\alpha,\beta}(f,x) = \sum_{k=0}^{m} \binom{m}{k} x^k (1-x)^{m-k} f\left(\frac{k+\alpha}{m+\beta}\right),$$

where $f \in C[0,1]$.

If $\alpha = 0$ and $\beta = 0$ then this operator reduces to the classical Bernstein operator. Such type of generalization got the attention of many mathematicians and special cases of these operators had been studied to investigate approximation properties of linear positive operators. There are numerous combinations and generalizations of the Stancu operators: Bernstein-Stancu operators, Schurer-Stancu operators, Durrmeyer-Stancu operators, Baskakov-Durrmeyer-Stancu operators, Stancu-Hurwitz operators, Kantorovich-Stancu operators and many others. Barbosu (2003) introduced Schurer-Stancu type operators and obtained its approximation properties. Abel and Gupta (2004) found the rate of convergence of Stancu Beta operators for functions of bounded variation. Stoica (2009) constructed Stancu type linear positive operators of approximation by using the Beta and the Gamma functions. Gupta and Yadav (2011) established better approximation by Stancu Beta operators using King’s approach. Verma et al. (2012) studied approximation properties of Baskakov–Durrmeyer–Stancu operators. Gupta et al. (2013) obtained simultaneous approximation for Szász–Mirakjan–Stancu–Durrmeyer operators.

It can be observed that most of the positive linear operators preserve the linear as well as constant functions. King (2003) made an approach to modify the Bernstein operators, so that the sequence of operators preserves constant and quadratic functions. The new sequence, thus obtained shows a better rate of convergence. This approach has been applied by many researchers to enhance the convergence estimates. (See, e.g. Gupta and Duman (2010), Gupta (2010), Duman and Ozarslan (2007), Duman et al. (2008), Gupta and Yadav (2011))

1.2. BASIC CONCEPTS, NOTATIONS AND CERTAIN DEFINITIONS

In this section, we give some basic definitions and results used in the theory of approximation which will be required for analysis in the present thesis.
1.2.1. Normed linear space

Let $K$ denote the field of scalars (real or complex) and let $L$ denote the linear space of functions $f$ defined for $x \in [a,b]$ over $K$. Then a mapping $\|\|: L \to K$ is called a norm on $L$ if the following conditions hold:

- a. $\|f\| > 0$ for all $f \in L$,
- b. $\|f\| = 0$ iff $f = 0$,
- c. $\|f + g\| \leq \|f\| + \|g\|$ for all $f, g \in L$,
- d. $\|kf\| \leq |k| \|f\|$ for all $k \in K$ and $f \in L$.

The space $(L, \|\|)$ is called a normed linear space.

For a mapping $f: [a,b] \to R$, the norm is defined as

$$\|f\| = \max_{a \leq x \leq b} |f(x)|.$$

A mapping $L: X \to Y$ is called a linear approximation method if the following conditions hold:

- a. $X$ is a linear space and $Y$ is a linear subspace of $X$,
- b. for every scalars $\mu$ and $\nu$ and for all $f, g \in X$, we have

$$L(\mu f + \nu g) = \mu L(f) + \nu L(g).$$

1.2.2. Linear positive operators

Let $f \in C[a, b]$, linear space of continuous real valued functions having the property $f(x) \geq 0$ for all $x \in [a, b]$. Then a mapping $L: C[a, b] \to C[a, b]$ is said to be a linear positive operator if it satisfies the following conditions:

- a. $L(\mu f + \nu g) = \mu L(f) + \nu L(g)$, for every scalars $\mu$ and $\nu$ and for all $f, g \in C[a, b]$ and $x \in [a,b]$
- b. $L(f, x) \geq 0$ for all $f \in C[a, b], x \in [a,b]$

Following mathematical structure is an example of linear positive operators on $C[a,b]$:

$$L_n(f, x) = \int_a^b \psi_n(x,y) f(y) dy,$$

where $\psi_n(x, y)$ is a positive kernel of $L_n(f, x)$. 
Its discrete structure is defined as

\[ L_n(f, x) = \sum_{i=1}^{n} \psi_{n,i}(x)f(y_i), \]

where \( \psi_{n,i}(x) \geq 0 \).

**1.2.3. Exponential type operators**

Let \( L_n(f, x) = \int_a^b \psi_n(x, y)f(y)dy \), be a positive operator on \( C[a,b] \) into \( C^\infty \) (class of infinitely differentiable functions on \([a,b]\)), \( \psi_n(x,y) \geq 0 \) is a kernel of distribution, and \( a, b \) may be \(-\infty, \infty\) respectively. Then \( L_n(., x) \) is said to be an exponential type operator if the following conditions hold

a. \( \int_a^b \psi_n(x, y)dy = 1 \),

b. \( \frac{\partial}{\partial x} \psi_n(x, y) = \frac{n(y-x)}{p(x)} \psi_n(x, y) \),

where \( p(x) \) is a polynomial of degree greater than one and \( p(x) > 0 \) on \([a, b]\).

**1.2.4. O-notation and o-notation**

Let \( \Phi(n) \) and \( \Psi(n) \) be functions of \( f \in N \) (the set of natural numbers) such that \( \Psi(n) > 0 \) then \( \Phi(n) = O(\Psi(n)) \) implies that there exists a positive constant \( \beta \) such that

\[ \lim_{n \to \infty} \frac{\Phi(n)}{\Psi(n)} \leq \beta. \]

But, if \( \lim_{n \to \infty} \frac{\Phi(n)}{\Psi(n)} = 0 \) then it is written as \( \Phi(n) = o(\Psi(n)) \).

**1.2.5. Modulus of continuity**

The \( m^{th} \) order modulus of continuity for a function continuous on \([a,b]\) is defined as

\[ \omega_m(f, \delta, [a,b]) = \sup \left\{ \Delta_m^h(f) \mid |h| \leq \delta; x, x+h \in [a,b] \right\}, \]
\( \Delta^n f(x) \) is the \( m \)th order forward difference of \( f(x) \) with step length \( h \). For \( m=1 \), we can write \( \omega(f, \delta), \omega_f(\delta) \) or \( \omega_1(f, \delta) \) in place of \( \omega_m(f, \delta) \).

**Properties of \( \omega(f, \delta) \)**

a. For some \( \mu \geq 0 \), we have \( \omega(f, \mu\delta) \leq (1 + \mu) \omega(f, \delta) \).

b. If \( n \) is a natural number, then \( \omega(f, n\delta) \leq n \omega(f, \delta) \).

c. If \( \omega(f, \delta) \) is an increasing function of \( \delta \) then for \( \delta_1 < \delta_2 \), it follows that \( \omega(f, \delta_1) \leq \omega(f, \delta_2) \).

d. \( f(x) \) is uniformly continuous on \( (a,b) \) if and only if

\[
\lim_{\delta \to 0} \omega(f, \delta) = 0.
\]

1.2.6. Lipschitz condition

A function \( f \in C[a,b] \) is said to satisfy Lipschitz condition of order \( \alpha > 0 \) if there exist a constant \( B > 0 \) and such that

\[
|f(x_i) - f(x_j)| \leq B |x_i - x_j|^\alpha \quad \text{for all } x_i, x_j \in [a,b].
\]

This class of functions is denoted by \( Lip \alpha \). Here it should be noticed that

\[
f \in Lip \alpha \quad \text{if and only if} \quad \omega(f, \delta) \leq B \delta^\alpha \quad \text{for } [a,b]
\]

This implies a function satisfying a Lipschitz condition is uniformly continuous. Also for \( \alpha > 1 \), if \( f \in Lip \alpha \) then \( f \) is a constant function.

1.2.7. Zygmund class

The class of functions \( f \) that satisfy the condition

\[
|f(x + \eta) + f(x - \eta - 2f(x))| \leq K\eta, \eta > 0
\]

is called Zygmund class. Its generalization is defined as:

Let \( \omega_r(f, \delta, a, b) \) be the \( r \)th order modulus of continuity. Then a generalized Zygmund class \( Liz(\alpha, k, a, b) \) is the class of functions \( f \) on \( [a, b] \) for which

\[
\omega_r(f, \delta, a, b) \leq B \delta^\alpha,
\]

where \( B \) is a constant.
1.2.8. $L_p$ space

If a function $f(x)$ is measurable on a finite interval $I$ then $|f|^p$ also measurable for each $p \in (0, \infty)$. Now $L_p(I)$ is defined as the class of all $p$-integrable functions over $I$ i.e.

$$L_p(I) = \left\{ f : \left[ \int_I |f|^p \right]^{1/p} < \infty \right\},$$

then norm of a function $f \in L_p(I)$ is defined by

$$\| f \|_{L_p} = \left[ \int_I |f|^p \right]^{1/p}, \quad 0 < p < \infty.$$

Also the space of complex valued measurable function that are essentially bounded is denoted and defined by

$$L_{\infty}(I) = \left\{ f : E_{\infty} \sup |f| < \infty \right\},$$

where $E_{\infty} \sup |f(x)| = \inf \left\{ A : |f(x)| \leq A \ \text{a.e. on} \ I \right\}$, $A$ is a constant, then norm in $L_{\infty}(I)$ is given by

$$\| f \|_{L_{\infty}} = \text{ess sup} |f|.$$

1.2.9. Characteristic function

A function $\Psi(x)$ on the interval $I$ defined as

$$\Psi(x) = \begin{cases} 1 & \text{if } x \in I \\ 0 & \text{if } x \notin I \end{cases}$$

is called characteristic function of $I$.

1.2.10. Taylor’s theorem for finite expansion of a function $f$

Let the function $f : [a, b] \to R$ be $k$-times differentiable at $x \in [a, b]$. Then we have

$$f(y) = \sum_{i=0}^{k} \frac{f^{(i)}(x)}{i!} (y-x)^i + \frac{\varepsilon(y,x)}{(k+1)!} (y-x)^k,$$

where $\varepsilon(y,x) \to 0$ as $y \to x$. 

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1.2.11. Signum function

The Signum function of a function $\Phi(x)$ is defined as

$$
\text{sgn } \Phi(x) = \begin{cases} 
1 & \text{if } \Phi(x) \geq 0 \\
-1 & \text{if } \Phi(x) < 0
\end{cases}.
$$

For $\Phi(x) = x$, we also write

$$
\text{sign}(x) = \begin{cases} 
1 & \text{if } x > 0 \\
0 & \text{if } x = 0 \\
-1 & \text{if } x < 0
\end{cases}.
$$

1.2.12. Cauchy-Schwarz inequality for integration and summation

Let $f$ and $g$ be scalar (real or complex) valued functions. Then, following inequalities hold for integration and summation respectively:

a. $\int |fg| \leq \left( \int |f|^2 \right)^{1/2} \left( \int |g|^2 \right)^{1/2},$

b. $\sum |f_i g_i| \leq \left( \sum |f_i|^2 \right)^{1/2} \left( \sum |g_i|^2 \right)^{1/2}.$

1.2.13. Holder’s inequality for integration and summation

Let $p$ and $q$ be two non-negative real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$. Then, following inequalities hold for integration and summation respectively:

a. $\int |fg| \leq \left( \int |f|^p \right)^{1/p} \left( \int |g|^q \right)^{1/q},$

b. $\sum |f_i g_i| \leq \left( \sum |f_i|^p \right)^{1/p} \left( \sum |g_i|^q \right)^{1/q}.$
1.2.14. Leibnitz theorem

Let $\Phi$ and $\psi$ be $k$-times differentiable functions. Then the $k^{\text{th}}$ derivative of their product is given by

$$D^k(\Phi \psi) = \sum_{i=0}^{k} \binom{k}{i} (D^{k-i}\Phi')(D^i\psi)$$

where $D^k$ denotes $k^{\text{th}}$ order differentiation operator.

1.2.15. The Beta function

Beta function is denoted and defined by

$$\beta(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1} \, dx = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}} \, dx \quad , \quad m, n > 0.$$  

1.2.16. The Gamma function

Gamma function is denoted and defined by

$$\Gamma(n) = \int_0^\infty x^{n-1}e^{-x} \, dx \quad , \quad n > 0.$$  

1.2.17. Lebesgue integrable functions

A bounded function $f(x)$ defined on a set $E$ of finite measure is said to be Lebesgue integrable over $E$, if

$$\sup_{\Phi \leq f} \int_E \Phi = \int_E f = \inf_{\psi \geq f} \int_E \psi$$

where $\Phi$ and $\Psi$ range over the set of all measurable functions defined on the set $E$. 

1.2.18. Lebesgue measurable functions

An extended real valued function $f$ (having its values in the extended real number system $\mathbb{R} \cup \{-\infty, \infty\}$) defined on a measurable set $E$ is said to be Lebesgue measurable or more briefly, measurable on $E$, if the set $E(f > \alpha) = \{x \in E : f(x) > \alpha\}$ is measurable for all real numbers $\alpha$.

1.2.19. Functions of bounded variation

A function $f$ defined on $[a, b]$ is said to be of bounded variation if there exists a positive constant $K$ such that for every partition $P = \{a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b\}$ of $[a, b]$, we have

$$\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \leq K.$$

Equivalently, $f$ is said to be of bounded variation if and only if it’s total variation

$$V(f) = \sup_{P} \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| < \infty.$$

1.2.20. $K$-functional

Let $C_0$ denote the class of continuous functions on the interval $[0, \infty)$ such that,

- It has a compact support,
- $C_0^{k} \subset C_0$ of $k$ times differentiable functions and
- $[a, b] \subset [0, \infty]$ and $H(a, b) = \{g \in C_0^{2k+2}, \text{supp } g \subset [a, b]\}$

Then the $K$-functional $K(\zeta, f, a, b)$ for $0 < \zeta \leq 1$ and $f \in C_0$ with $\text{supp } f \subset [a, b]$ is defined by

$$K(\zeta, f, a, b) = \inf_{g} \{\|f - g\| + \zeta (\|g\| + \|g^{(2k+2)}\|)\},$$

Where $\|\cdot\|$ denotes the sup-norm on $(0, \infty)$. For $0 < \mu < 2$, $C_0(\mu, k + 1, a, b)$ denotes the set of function $f$ for which

$$\sup_{0 < \zeta \leq 1} \zeta^{-\mu/2} K(\zeta, f, a, b) < B,$$

for some constant $B$. 

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1.2.21. Steklov mean

When $r^{th}$ modulus of smoothness has to be estimated in terms of $r^{th}$ derivative of $f$ then usually mean of integrals i.e. Steklov mean is used, which is defined as:

Let $C_r[0, \infty] = \{ f \in C[0, \infty] : |f(t)| \leq M |g(t)|^r, M > 0 \}$, where $g(t)$ is some growth function with norm $\| \cdot \|_r$ on $C_r[0, \infty]$ defined as

$$\|f\|_r = \sup_{0 \leq t \leq \infty} |f(t)|(g(t))^{-r}.$$

Then for $0 < a < a_1 < b_1 < b < \infty$, for sufficiently small $\eta > 0$ Steklov mean denoted by $f_{\eta,m}$ of order $m$ corresponding to $f \in C_r[0, \infty]$, and $h \in [a, b]$ is defined as

$$f_{\eta,m}(t) = \frac{1}{\eta^m} \int_{-\eta/2}^{\eta/2} \int_{-\eta/2}^{\eta/2} (f(t) - \Delta_h^m f(t)) dt_1 \ldots dt_1,$$

where $t = \frac{1}{m} \sum_{i=1}^m h_i$, and $\Delta_h^m$ is the $m^{th}$ order forward difference operator with step length $h$. Steklov mean is a linear approximating function and satisfy the following properties:

a. $f_{\eta,m}$ has continuous derivatives up to order $m$ over $[a,b]$.

b. $\|f^{(r)}_{\eta,m}\|_{C[a_1,b_1]} \leq K_1 \eta^{-r} \omega_r(f, \eta, a_1, b_1), r = 1, 2, \ldots, m$.

c. $\|f - f_{\eta,m}\|_{C[a_1,b_1]} \leq K_2 \omega_m(f, \eta, a, b)$.

d. $\|f_{\eta,m}\|_{C[a_1,b_1]} \leq K_3 \|f\|_r$.

where by $K_i$ we mean certain constants not same at each occurrence and are independent of $f$ and $\eta$. $\omega_m(f, \eta, a, b)$ is the modulus of continuity of order $m$ for $f$.

1.2.22. Hypergeometric function

The hypergeometric function is given by

$$\, _2F_1 (u, v; w; x) = \sum_{0}^{\infty} \frac{(u)_k (v)_k}{(w)_k k!} x^k,$$

and confluent hypergeometric function is defined by

$$\, _1F_1 (u; v; x) = \sum_{0}^{\infty} \frac{(u)_k}{(v)_k k!} x^k.$$
where the Pochhammer symbol \((a)_k\) denotes increasing factorial given as 

\[(a)_k = a (a+1)(a+2) \ldots \ldots \ldots (a+k-1).
\]

1.2.23. Ditzian-Totik modulus

The Ditzian-Totik modulus of second order is defined as

\[\omega^2_{\psi}(f, \delta) = \sup_{|h| \leq \delta} \sup_{x \pm \delta} |f(x - \psi h) - 2f(x) + f(x + \psi h)|\]

In which \(\psi: [0,1] \rightarrow \mathbb{R}\) is step weight function and is given as \(\psi = \sqrt{x(1-x)}\).

1.3 LINEAR POSITIVE OPERATORS STUDIED IN THE THESIS

The present work of the thesis is devoted to study of approximation properties of mixed summation-integral type operators. The following sequences of positive linear operators are introduced (or studied) in the present thesis.

1. Baskakov-Durrmeyer operators

\[R_n^x(f, x)(x) = a^{2n-1} (n-1) \sum_{k=0}^{\infty} p_{n,k}(x) \int_0^1 p_{n,k}(t) f(t) dt,
\]

where for \(a > 0\), the generalized Baskakov basis function is defined as

\[p_{n,k}(x) = \binom{n+k-1}{k} x^k (a + x)^{-n-k}.
\]

2. Generalized Bernstein-Kantorovich type operators

\[R_n^f(x) = (n+1) \sum_{k=0}^{n} h_{n,r,k}(x) \int_{k/(n+1)}^{(k+1)/(n+1)} f(t) dt\]

where the generalized Bernstein basis function is defined as

\[h_{n,r,k}(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{x(x+k\alpha)^{k-1}(1-x+r/n+(n-k)\alpha)^{n-k}}{\left(1+r/n+n\alpha\right)^n}.
\]
3. Baskakov-Kantorovich type operators

\[ K_n^{a,c}(f, x) = n \sum_{k=0}^{\infty} b_n^{a,c}(x) \int_{k/n}^{(k+1)/n} f(t) dt, \]

where the generalized Baskakov basis function is given by

\[ b_n^{a,c}(x) = e^{-a\alpha(1+\alpha)} \frac{1}{k!} \sum_{i=0}^{k} \binom{k}{i} (n/\alpha) \cdot a^{k-i} \cdot \frac{(\alpha x)^i}{(1+\alpha)^{\alpha x+k}}. \]

4. Szász-Mirakyan-Baskakov-Stancu type operators

\[ S_{n,\alpha,\beta}(f, x) = (n-1) \sum_{k=0}^{\infty} \varphi_{n,k}(x) \int_{0}^{\infty} \psi_{n,k}(y) f\left(\frac{nt + \alpha}{n + \beta}\right) dt, \quad x \in [0, \infty), \]

where \( \alpha \) and \( \beta \) are some non zero parameter and the Szász and Baskakov basis function are given as

\[ \varphi_{n,k}(x) = e^{-\alpha x} \frac{(nx)^k}{k!}, \quad \psi_{n,k} = \frac{(n)_{k}}{k!} \frac{t^k}{(1+t)^{n+k}}. \]

where \((n)_{k}\) is is defined as \((n)_{k} = n(n+1)(n+2)\ldots(n+k-1) \) and \((n)_{0} = 1.\)

5. General Summation-integral type operators

\[ G_{\rho,n,c}(f, x) = \sum_{k=1}^{\infty} p_{n,k}(x, c) \int_{0}^{\infty} q_{\rho,n+c,k-1}(t, c) f\left[\frac{n\rho - c}{n\rho}\right] dt + p_{n,0}(x, c) f(0), \]

where the generalized basis functions are given by

\[ p_{n,k}(x) = \frac{(-x)^k}{k!} \varphi_{n,c}^{(k)}(x) \text{ with } \varphi_{n,c}(x) = \begin{cases} e^{-\alpha x}, & c = 0 \\ (1+\alpha x)^{-n/c}, & c \in N \\ (1-x)^n, & c = -1 \end{cases} \]

and

\[ q_{\rho,n+c,k-1}(t, c) = \begin{cases} \frac{n\rho e^{-n\rho} (n\rho t)^{k-1}}{\Gamma(k\rho)}, & c = 0 \\ e^{\rho t} t^{k-1}, & c \in N \\ B\left(\frac{n\rho}{c}, k\rho\right) (1+ct)^{n\rho/c^{k\rho}}, & c \in N \end{cases} \]
6. Beta-Szász Operator

\[ B_n^\beta(f, x) = \sum_{r=0}^{\infty} b_{n,r}(\alpha(x)) \int_{0}^{\infty} s_{n,r}(y) f(y) dy, \]

where \( \alpha(x) = \frac{nx-1}{n+1} \) and Beta and Szász basis functions are given as

\[ b_{n,r}(x) = \frac{1}{B(r+1,n)} \frac{x^r}{(1+x)^{n+r+1}} \text{ and } S_{n,r}(y) = e^{-ny} \frac{(ny)^r}{r!}. \]

7. Genuine integral type Lupaş-Beta operators

\[ J_n(f, x) = \sum_{k=0}^{\infty} l_{n,k}(x) \int_{0}^{\infty} b_{n,k-1}(t) f(t) dt + 2^{-nx} f(0), \quad x \geq 0, \]

where Lupaş and Beta basis function are given as

\[ l_{n,k}(x) = 2^{-nx} \frac{(nx)_k}{k!} 2^k, \quad b_{n,k-1}(t) = \frac{1}{B(k,n+1)} \frac{t^{k-1}}{(1+t)^{k+n+1}}. \]

1.4 CONTENT OF THE THESIS

The present thesis concerns itself with the study of approximation properties of summation and integral type positive linear operators. We discuss the convergence behavior of some sequences of linear positive operators in terms of moments, point wise convergence, asymptotic expansion, and upper bound for the estimate of error function. The content of the thesis is divided into nine chapters.

Chapter 1: This chapter presents an introductory background of approximation theory and defines the problem of approximation. We emphasize the related work done and contributions in the theory of approximation made by several eminent researchers. Next we include basic notations and definitions, which are important for analysis of approximation properties. Further, the sequences of positive linear operators studied in the present thesis are mentioned. The last segment of the chapter conveys a succinct summary of the content of the thesis.

Chapter 2: In this chapter, we propose a certain modified family of summation and integral type operators to approximate the functions integrable on \([0, \infty)\). This family is actually a Durrmeyer kind integral variation of Baskakov operators. We consider the class of the entire Lebesgue measurable functions \( f(x) \) on \([0, \infty)\) such that \( \int_{0}^{\infty} \frac{|f(x)|}{(a+nx)^n} dx < \infty \), where \( n \) is some positive integer and \( a>0 \). Firstly, central
moments for the defined operators are determined. Next, we obtain a Voronovskaja type asymptotic expansion for the operators by utilizing moment estimates. At last of the chapter, an upper bound of error estimate in simultaneous approximation by means of the modulus of continuity of first order is obtained.

Chapter 3: In this chapter, a modification of generalized Bernstein polynomial due to Kantorovich (1930) is introduced to make it feasible to approximate Lebesgue integrable functions in $L_1$-norm. We extend the result due to Lorentz (1953) and Voronowskaja (1932) for Lebesgue integrable functions in $L_1$-norm by the defined polynomial on the interval $[0, 1 + r/n]$.

Chapter 4: In this chapter, we propose and study certain modified Baskakov-Kantorovich type operators. We obtain asymptotic expansion by means of moments estimates for the operators. Next result determine error estimation formula in terms of second order of the modulus of continuity by means of K-functional and Steklov mean. The last segment of the chapter considers the functions with derivative of bounded variation and obtains the rate of convergence for the defined operators.

Chapter 5: In this chapter, Stancu type generalization of Szász-Mirakyan-Baskakov kind operators is considered. First, we make use of hypergeometric series and obtain the moments for the defined operators, which can be linked to Laguerre polynomials. In the next section, the point wise convergence and asymptotic expansion are established for the defined operator. Further, order of approximation in terms of higher order modulus of continuity of function is obtained in simultaneous approximation for the defined operators. The Steklov mean, technique of linear approximating method is used to find the estimates.

Chapter 6: In this chapter, the convergence characteristics of a modified sequence of positive linear operators, known as Srivastava-Gupta operators introduced by Srivastava and Gupta (2003), is analyzed in simultaneous approximation. We estimate central moments for the defined operators. Subsequently, some direct theorems, including point wise convergence, asymptotic expansion and error estimation in terms of the modulus of continuity for the defined operators are acquired.

Chapter 7: In this chapter, we construct a sequence of Beta-Szász type operators so as to make it to preserve linear functions utilizing an approach made by King (2003). We derive some direct theorems for the defined operators which include point wise convergence using the Korovkin type theorem and an estimate of error in approximation of the function expressed in terms of first order modulus of continuity. In the last section of the chapter, we claim a superior rate of convergence for the modified operators.
Chapter 8: In this chapter, we study the convergence properties of genuine Lupaş-Beta operators of integral type. We find moments estimates required for the analysis up to order six by using factorial polynomial and an elementary hypergeometric function. Subsequently, a quantitative asymptotic formula and some direct estimates in terms of modulus of continuity due to Ditzian-Totik are obtained. At the end, we mention certain results on the weighted modulus of continuity due to Păltănea (2011) for the concerned operators.

Finally chapter nine of the thesis presents the future scope of the work. It presents some problems that can be considered for further analysis.