CHAPTER 8

APPROXIMATION PROPERTIES OF INTEGRAL TYPE LUPAŞ-BETA OPERATORS

8.1. INTRODUCTION

In the year 1995 Lupaş (1995) proposed yet another important discrete operator as

\[ L_n(f, x) = \sum_{k=0}^{\infty} l_{n,k}(x) f(k/n), x \in [0, \infty) \]  \hspace{1cm} (8.1)

where

\[ l_{n,k}(x) = 2^{-nx} \frac{(nx)^k}{k!2^k}. \]

Agratini (1999) considered the Lupaş-Kantorovich operators defined as

\[ K_n(f, x) = \sum_{k=0}^{\infty} l_{n,k}(x) \int_{k/n}^{(k+1)/n} f(t)dt, \]

where \( l_{n,k}(x) \) is as defined in Eqn. (8.1). He obtained some direct results and also proposed Lupaş-Durrmeyer operators. But for Durrmeyer variant, the approximation properties are not easily obtained. Aral and Gupta (2016) obtained direct estimates for Lupaş-Durrmeyer operators.

Abel and Ivan (2007) studied the general form of the operators defined in Eqn. (8.1) and obtained the complete asymptotic expansion of these operators. Several modifications of positive linear operator have been introduced to study the convergence behaviour in different directions. Govil et al. (2013) considered the hybrid operators by taking the weights of Szász basis functions to modify the operators defined in Eqn. (8.1). Further to extend the studies, Gupta and Yadav (2014) considered other hybrid operators by taking weights of Beta basis functions. But the operators considered by Gupta and Yadav (2014) reproduce only the constant functions. Later Gupta et al. (2014) considered the following form of hybrid operators, which preserve constant as well as linear functions.
\[ J_n(f, x) = \sum_{k=1}^{\infty} l_{n,k}(x) \int_0^\infty b_{n,k-1}(t) f(t) dt + 2^{-nx} f(0), x \geq 0 \]

where
\[ l_{n,k}(x) = 2^{-nx} \frac{(nx)_k}{k! 2^x}, \quad b_{n,k-1}(t) = \frac{1}{B(k,n+1)} \frac{t^{k-1}}{(1+t)^{k+n+1}} \]

and \( B(m,n) \) being the Beta function. In the present Chapter, We intend to study the operators \( J_n(f, x) \). We obtain a result for quantitative asymptotic formula in terms of weighted modulus of continuity and a direct estimate in terms of Ditzian-Totik modulus of continuity. Next, we intend to show the validity of the operators \( J_n(f, x) \) for the weighted modulus of continuity due to Păltănea (2011). These results will appear in Gupta et al. (2017).

### 8.2. MOMENTS

Gupta et al. (2014) obtained the moments using hypergeometric function given as
\[ J_n(t^m, x) = \frac{r!(n-m)!(nx)}{n!} \text{F}(nx+1,1-m;2;1). \]

We can apply some other methods to find the moments. In this section, we apply the concept of factorial polynomials to find the moments, defined as
\[ k^{(m)} = k(k-1)(k-2)\ldots(k-m+1) \]

and the elementary hypergeometric functions \( _r F_0(a;-;x) \).

**Lemma 8.2.1.** The moments for the operators \( J_n(f, x) \) are defined as
\[
\begin{align*}
J_n(1, x) &= 1, \\
J_n(t, x) &= x, \\
J_n(t^2, x) &= \frac{nx^2 + 3x}{n-1}, \\
J_n(t^3, x) &= \frac{n^2x^3 + 9nx^2 + 14x}{(n-1)(n-2)}, \\
J_n(t^4, x) &= \frac{n^3x^4 + 18n^2x^3 + 83nx^2 + 90x}{(n-1)(n-2)(n-3)}, \\
J_n(t^5, x) &= \frac{n^4x^5 + 30n^3x^4 + 275n^2x^3 + 870nx^2 + 744x}{(n-1)(n-2)(n-3)(n-4)}, \text{ and} \\
J_n(t^6, x) &= \frac{n^5x^6 + 45n^4x^5 + 685n^3x^4 + 4275n^2x^3 + 10474nx^2 + 7560x}{(n-1)(n-2)(n-3)(n-4)(n-5)}. \\
\end{align*}
\]
Proof: By definition of \( b_{n,k-1}(t) \), we have
\[
\int_0^\infty b_{n,k-1}(t)t^m dt = \int_0^\infty \frac{1}{B(k,n+1)} \frac{t^{k+m-1}}{(1+t)^{k+n+1}} dt
\]
\[
= \frac{(k+m-1)!.(n-m)!}{n!.(k-1)!}.
\]
Thus using the above identity and the fact that
\[
_1F_0(a;-;z) = \sum_{k=0}^{\infty} (a)_k \frac{z^k}{k!} = (1 - z)^{-a}, |z| < 1,
\]
we get
\[
J_n(1,x) = \sum_{k=1}^{\infty} J_{n,k}(x) + J_{n,0}(x) = \sum_{k=0}^{\infty} J_{n,k}(x)
\]
\[
= 2^{-nx} \sum_{k=0}^{\infty} \frac{(nx)_k}{k!.2^k} = 2^{-nx} J_0^n(\frac{nx^-; \frac{1}{2}}{2})
\]
\[
= 2^{-nx} \left(1 - \frac{1}{2}\right)^{-nx} = 1.
\]
Now
\[
J_n(t,x) = \sum_{k=1}^{\infty} J_{n,k}(x) \frac{k}{n} = 2^{-nx} n \sum_{k=0}^{\infty} \frac{(nx)_k}{k!(k-1).2^k}
\]
\[
= 2^{-nx} n \sum_{k=0}^{\infty} \frac{(nx)_{k+1}}{k! .2^{k+1}} = 2^{-nx-1} n \sum_{k=0}^{\infty} \frac{nx(nx+1)_k}{k! .2^k}
\]
\[
= x.2^{-nx-1} J_0^n(nx+1;-; \frac{1}{2})
\]
\[
= x.2^{-nx-1} \left(1 - \frac{1}{2}\right)^{-nx-1} = x.
\]
Writing \( k(k+1) \) in terms of factorial polynomials i.e. \( k^2 + k = k^{(2)} + 2k^{(1)} \) and using \( (nx)_{k+2} = nx(nx+1)(nx+2)_k \), we have
\[
J_n(t^2,x) = \sum_{k=1}^{\infty} J_{n,k}(x) \frac{k^2 + k}{n(n-1)} = \sum_{k=1}^{\infty} J_{n,k}(x) \frac{k^{(2)} + 2k^{(1)}}{n(n-1)}
\]
\[
\text{Also, we have}
\]
\[
J_a(t^2, x) = \sum_{k=1}^{\infty} j_{n,k}(x) \frac{k^3 + 3k^2 + 2k}{n(n-1)(n-2)}
\]
\[
= \sum_{k=1}^{\infty} j_{n,k}(x) \frac{k(3) + 6k(2) + 6k(1)}{n(n-1)(n-2)}
\]
\[
= \frac{2^{-n} x}{n(n-1)(n-2)} \left[ \sum_{k=3}^{\infty} \frac{(nx)_k}{(k-3)!2^k} + 6 \sum_{k=2}^{\infty} \frac{(nx)_k}{(k-2)!2^k} + 6 \sum_{k=1}^{\infty} \frac{(nx)_k}{(k-1)!2^k} \right]
\]
\[
= \frac{2^{-n} x}{n(n-1)(n-2)} \left[ \sum_{k=0}^{\infty} \frac{(nx)_{k+3}}{k!2^{k+3}} + 6 \sum_{k=0}^{\infty} \frac{(nx)_{k+2}}{k!2^{k+2}} + 6 \sum_{k=0}^{\infty} \frac{(nx)_{k+1}}{k!2^{k+1}} \right]
\]
\[
= \frac{2^{-n-3} nx(nx+1)(nx+2)}{n(n-1)(n-2)} \sum_{k=0}^{\infty} \frac{(nx+3)_k}{k!2^k} + 6 \frac{2^{-n-2} nx(nx+1)}{n(n-1)(n-2)} \sum_{k=0}^{\infty} \frac{(nx+2)_k}{k!2^k}
\]
\[
+ 6 \frac{2^{-n-1} nx}{n(n-1)(n-2)} \sum_{k=0}^{\infty} \frac{(nx+1)_k}{k!2^k}
\]
\[
= \frac{x(nx+1)(nx+2)}{(n-1)(n-2)} + \frac{6x(nx+1)}{(n-1)(n-2)} + \frac{6x}{(n-1)(n-2)}.
\]
Next, using \( k^4 + 6k^3 + 11k^2 + 6k = k^{(4)} + 12k^{(3)} + 36k^{(2)} + 18k^{(1)} \), we get

\[
J_n(t^4, x) = \sum_{k=1}^{\infty} \int_{[t, x]} \frac{k^4 + 6k^3 + 11k^2 + 6k}{n(n-1)(n-2)(n-3)} \, dx
\]

\[
= \frac{x(nx+1)(nx+2)(nx+3)}{(n-1)(n-2)(n-3)} + \frac{12x(nx+1)(nx+2)}{(n-1)(n-2)(n-3)}
\]

\[
+ \frac{36x(nx+1)}{(n-1)(n-2)(n-3)} + \frac{18x}{(n-1)(n-2)(n-3)}.
\]

Further using

\( k^5 + 10k^4 + 35k^3 + 50k^2 + 24k = k^{(5)} + 20k^{(4)} + 120k^{(3)} + 240k^{(2)} + 120k^{(1)} \), we have

\[
J_n(t^5, x) = \sum_{k=1}^{\infty} \int_{[t, x]} \frac{k^5 + 10k^4 + 35k^3 + 50k^2 + 24k}{n(n-1)(n-2)(n-3)(n-4)} \, dx
\]

\[
= \frac{x(nx+1)(nx+2)(nx+3)(nx+4)}{(n-1)(n-2)(n-3)(n-4)} + \frac{20x(nx+1)(nx+2)(nx+3)}{(n-1)(n-2)(n-3)(n-4)}
\]

\[
+ \frac{120x(nx+1)(nx+2)}{(n-1)(n-2)(n-3)(n-4)} + \frac{240x(nx+1)}{(n-1)(n-2)(n-3)(n-4)}
\]

\[
+ \frac{120x}{(n-1)(n-2)(n-3)(n-4)}.
\]

Finally using

\( k^6 + 15k^5 + 85k^4 + 225k^3 + 274k^2 + 120k = k^{(6)} + 30k^{(5)} + 300k^{(4)} + 1200k^{(3)} + 1800k^{(2)} + 720k^{(1)} \),

we get

\[
J_n(t^6, x) = \sum_{k=1}^{\infty} \int_{[t, x]} \frac{k^6 + 15k^5 + 85k^4 + 225k^3 + 274k^2 + 120k}{n(n-1)(n-2)(n-3)(n-4)(n-5)} \, dx
\]

\[
= \frac{x(nx+1)(nx+2)(nx+3)(nx+4)(nx+5)}{(n-1)(n-2)(n-3)(n-4)(n-5)} + \frac{30x(nx+1)(nx+2)(nx+3)(nx+4)}{(n-1)(n-2)(n-3)(n-4)(n-5)}
\]

\[
+ \frac{300x(nx+1)(nx+2)(nx+3)}{(n-1)(n-2)(n-3)(n-4)(n-5)} + \frac{1200x(nx+1)(nx+2)}{(n-1)(n-2)(n-3)(n-4)(n-5)}
\]

\[
+ \frac{1800x(nx+1)}{(n-1)(n-2)(n-3)(n-4)(n-5)} + \frac{720x}{(n-1)(n-2)(n-3)(n-4)(n-5)}.
\]
Remark 8.2.1. If \( M_{n,m}(x) = J_n((t-x)^m, x) \), then by simple computation and Lemma 8.2.1, we have

\[
M_{n,1}(x) = 0,
\]

\[
M_{n,2}(x) = \frac{x(x+3)}{n-1},
\]

\[
M_{n,4}(x) = J_n((t-x)^4, x) = J_n(t^4, x) - 4xJ_n(t^3, x) + 6x^2J_n(t^2, x) - 4x^3J_n(t, x) + x^4
\]

\[
= \frac{3(n+6)x^4 + 18(n+6)x^3 + (27n+168)x^2 + 90x}{(n-1)(n-2)(n-3)}.
\]

\[
M_{n,6}(x) = J_n(t^6, x) - 6xJ_n(t^5, x) + 15x^2J_n(t^4, x) - 20x^3J_n(t^3, x) + 15x^4J_n(t^2, x) + x^6
\]

\[
= J_n(t^6, x) - 6xJ_n(t^5, x) + 15x^2J_n(t^4, x) - 20x^3J_n(t^3, x) + 15x^4J_n(t^2, x) - 5x^6
\]

\[
= \frac{n^5x^6 + 45n^4x^5 + 685n^3x^4 + 4275n^2x^3 + 10474nx^2 + 7560x}{(n-1)(n-2)(n-3)(n-4)(n-5)}
\]

\[
- 6xn^4x^3 + 30n^3x^4 + 275n^2x^3 + 870nx^2 + 744x
\]

\[
+ 15x^4n^3x^3 + 18n^2x^4 + 83nx^3 + 90x
\]

\[
- 20x^3n^2x^3 + 9nx^4 + 14x
\]

\[
+ 15x^4nx^2 + 3x
\]

\[
= \frac{1}{(n-1)(n-2)(n-3)(n-4)(n-5)}[(15n^2 + 430n + 600)x^6
\]

\[
+ (135n^2 + 3870n + 5400)x^5 + (405n^2 + 11740n + 16800)x^4
\]

\[
+ (405n^2 + 13950n + 27000)x^3 + (6010n + 22320)x^2 + 7560x].
\]

We can conclude here that

\[
M_{n,m}(x) = O(n^{-(m+1)/2}).
\]
8.3. DIRECT ESTIMATES

Let \( C_2 \{0, \infty\} = C[0, \infty) \cap W_2 \{0, \infty\} \), where \( W_2 \{0, \infty\} \) is the set of all functions \( f \) defined on \( \mathbb{R}^+ \) such that the condition \( |f(x)| \leq B_j (1 + x^2) \) should be satisfied by \( f \), where \( B_j \) is any constant which depends only on \( f \), but independent of \( x \). By \( C_2 \{0, \infty\} \), we denote subspace of all continuous functions \( f \in W_2 \{0, \infty\} \) for which \( \lim_{x \to \infty} \frac{f(x)}{1 + x^2} \) is finite.

The weighted modulus of continuity \( \Omega(f, \delta) \) defined on an infinite interval \( \mathbb{R}^+ = [0, \infty) \) (see Acar et al. (2016)) is given by

\[
\Omega(f, \delta) = \sup_{|x|, |\delta|, x \in \mathbb{R}^+} \frac{|f(x + h) - f(x)|}{(1 + h^2)(1 + x^2)} \quad \text{for each } f \in C_2 \{0, \infty\}.
\]

Some basic properties of \( \Omega(f, \delta) \) are mentioned in the following Lemma 8.3.1 (see Acar et al. (2016)).

**Lemma 8.3.1.** Let \( f \in C_2 \{0, \infty\} \). Then,

a. \( \Omega(f, \delta) \) is a monotonically increasing function of \( \delta \), \( \delta \geq 0 \).

b. For every \( f \in C_2 \{0, \infty\} \), \( \lim_{\delta \to 0} \Omega(f, \delta) = 0 \).

c. For each \( \lambda > 0 \), we have \( \Omega(f, \lambda \delta) \leq 2(1 + \lambda)(1 + \delta^2) \Omega(f, \delta) \)

We now estimate the following quantitative Voronovskaja type asymptotic formula:

**Theorem 8.3.1.** Let \( f^* \in C_2 \{0, \infty\} \), and \( x > 0 \). Then, we have

\[
\left| J_n(f, x) - f(x) - \frac{x(x+3)}{2(n-1)} f^*(x) \right| \leq 8(1 + x^2) O(n^{-1}) \Omega \left( f^*, \frac{1}{\sqrt{n}} \right).
\]

**Proof:** By the Taylor’s formula, there exist \( \eta \) lying between \( x \) and \( y \) such that

\[
f(y) = f(x) + f'(x)(y-x) + \frac{f''(x)}{2} (y-x)^2 + h(y,x)(y-x)^3,
\]

where

\[
h(y,x) = \frac{f'''(\eta)}{6} (y-x)^3.
\]
where
\[ h(y, x) := \frac{f''(\eta) - f''(x)}{2} \]
and \( h \) is a continuous function which vanishes at 0. Applying the operator \( J_n \) to above equality and using Remark 8.2.1, we get
\[ J_n(f, x) - f(x) = \frac{f''(x)}{2} \left[ \frac{x(x + 3)}{n-1} \right] + J_n(h(y, x)(y-x)^2, x) \]
Also, we can write that
\[ J_n(f, x) - f(x) - \frac{f''(x)}{2} \left( \frac{x(x + 3)}{n-1} \right) \leq J_n \left( \| h(y, x)(y-x)^2, x \| \right) \]
To estimate last inequality using Lemma 8.3.1 and the inequality \( |\eta - x| \leq |y - x| \), we can write that
\[ |h(y, x)| \leq \left( 1 + (y-x)^2 \right) \left( 1 + \frac{|y-x|}{\delta} \right) \left( 1 + \delta^2 \right) \Omega(f'', \delta) \]
Also,
\[ |h(y, x)| \leq \begin{cases} 2\left( 1 + x^2 \right) \left( 1 + \delta^2 \right)^2 \Omega(f'', \delta), & |y-x| < \delta \\ \left( 1 + (y-x)^2 \right) \left( 1 + \frac{|y-x|}{\delta} \right) \left( 1 + \delta^2 \right) \Omega(f'', \delta), & |y-x| \geq \delta \end{cases} \]
Now choosing \( \delta < 1 \), we have
\[ |h(y, x)| \leq 2\left( 1 + x^2 \right) \left( 1 + \frac{|y-x|^4}{\delta^4} \right) \left( 1 + \delta^2 \right)^2 \Omega\left( f'', \delta \right) \]
\[ \leq 8\left( 1 + x^2 \right) \left( 1 + \frac{|y-x|^4}{\delta^4} \right) \Omega\left( f'', \delta \right) \]
Using Remark 8.2.1, we obtain
\[ J_n \left( |h(y, x)|(y-x)^2, x \right) = 8\left( 1 + x^2 \right) \Omega\left( f'', \delta \right) \left[ J_n \left( (t-x)^2, x \right) + \frac{1}{\delta^4} J_n \left( (t-x)^6, x \right) \right] \]
\[ = 8\left( 1 + x^2 \right) \Omega\left( f'', \delta \right) \left[ O(n^{-1}) + \frac{1}{\delta^4} O(n^{-3}) \right]. \]
Finally, choosing \( \delta = 1/\sqrt{n} \), we have
\[ J_n \left( |h(y, x)|(y-x)^2, x \right) = 8\left( 1 + x^2 \right) \Omega\left( f'', \delta \right) O\left( n^{-1} \right) \]
This completes the proof of the theorem.
8.4. DIRECT RESULT IN TERMS OF DT-MODULUS OF CONTINUITY

By $C_\beta[0,\infty)$, we denote the class of all real valued continuous and bounded functions $f$ on $[0,\infty)$. The second order Ditzian-Totik modulus of smoothness is defined by:

$$\omega_2^2(f, \delta) = \sup_{0<s<h} \sup_{x \pm h \varphi(x) \in [0,\infty)} |f(x+h\varphi(x)) - 2f(x) + f(x-h\varphi(x))|,$$

$$\varphi(x) = \sqrt{x(x+3)}, \ x \geq 0.$$  

The corresponding $K$-functional is:

$$K_{2,\varphi}(f, \delta^2) = \inf_{h \in W_\varphi^2(f)} \{ \| f - h \| + \delta^2 \| \varphi^2h'' \| \},$$

where $W_\varphi^2(\varphi) = \{ h \in C_\varphi[0,\infty); \ h \in AC_{loc}[0,\infty); \varphi^2h'' \in C_\beta[0,\infty] \}$. By theorem 2.1.1 of Ditzian and Totik (1987), it follows that

$$C^{-1}\omega_2^2(f, \delta) \leq K_{2,\varphi}(f, \delta^2) \leq C\omega_2^2(f, \delta),$$

for some absolute constant $C > 0$.

Our following direct result is in terms of Ditzian-Totik modulus of smoothness:

**Theorem 8.4.1.** If $f \in C_\beta[0,\infty)$ and $n \in \mathbb{N}$, then we have the following inequality:

$$\| J_n(f, x) - f(x) \| \leq 4\omega_2^2(f, \frac{1}{\sqrt{n}}).$$

**Proof:** We set $\varphi(x) = \sqrt{x(x+3)}$, $W_\varphi^2[0,\infty) = \{ g \in AC_{loc}[0,\infty); \varphi^2g'' \in C_\beta[0,\infty] \}$ then,

$$\frac{|y-u|}{\varphi''(u)} \leq \frac{|y-x|}{\varphi''(x)} \text{ for } x \leq u \leq y$$

and for $g \in W_\varphi^2[0,\infty)$, by Taylor’s formula, we have:

$$g(y) = g(x) + g'(x)(y-x) + \int_x^y g''(u)(y-u)du.$$  

Applying the operator $J_n$ to above equality and then taking modulus, we get:

$$|J_n(g, x) - g(x)| \leq J_n(\int_x^y (y-u)g''(u)du, x)$$

$$\leq \| \varphi^2g'' \| J_n(\frac{(y-x)^2}{x(x+3)}, x) = \frac{1}{n-1}\| \varphi^2g'' \|.$$  

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Now for $f \in C_2[0, \infty)$, we have

\[
|J_n(f, x) - f(x)| = |J_n(f - g, x) - (f - g)(x)| + |J_n(g, x) - g(x)|
\]

\[
\leq 4\|f - g\| + \frac{1}{n-1}\|\varphi^2 g\|
\]

\[
\leq 4\left\{\|f - g\| + \frac{1}{n-1}\|\varphi^2 g\|\right\}.
\]

Hence, by definition of $K_{2\varphi}(f, \delta^2)$, we have the inequality:

\[
\|J_n(f, x) - f(x)\| \leq 4K_{2\varphi}\left(f, \frac{1}{n}\right) \leq 4\omega^2_{\varphi}\left(f, \frac{1}{\sqrt{n}}\right).
\]

Thus the error estimate is completed in terms of Ditzian-Totik modulus of smoothness.

\section*{8.5. APPLICATIONS TO WEIGHTED MODULUS OF CONTINUITY}

Päätänea in (2011) considered the weighted modulus of continuity $\omega_{\varphi}(f; h)$, defined by

\[
\omega_{\varphi}(f; h) = \sup\left\{|f(x) - f(y)| : x \geq 0, y \geq 0, |x - y| \leq h \varphi\left(\frac{x + y}{2}\right)\},\ h \geq 0
\]

where the weight function is taken as $\varphi(x) = \frac{\sqrt{x}}{1 + x^m}, x \in [0, \infty), m \in \mathbb{N}, m \geq 2$. We consider here those functions, for which weighted modulus of continuity satisfies the property

\[
\lim_{h \to 0}\omega_{\varphi}(f; h) = 0.
\]

It is easy to verify that this property is fulfilled by an algebraic polynomial of degree $\leq m$. This follows from Theorem 2 in Päätänea (2011), which states that $\lim_{h \to 0}\omega_{\varphi}(f; h) = 0$ whenever the function $f * e_2$ is uniformly continuous on $[0, 1]$ and the function $f * e_\nu, \nu = \frac{2}{2m + 1}$ is uniformly continuous on $[1, \infty)$, where $e_\nu(x) = x^\nu, x \geq 0$. By $P$, we refer to the subspace of $C[0, \infty)$ which contains the polynomials.
Let us denote by $W_\phi[0,\infty)$ the subspace of all real functions defined on $[0,\infty)$, for which the two conditions mentioned above hold true. Very recently Tachev and Gupta (2016) obtained a quantitative estimates in terms of the above Páltánea’s modulus of continuity. The operators discussed here also preserve linear functions, so can be applied with this modulus of continuity. By the same arguments as in Tachev and Gupta (2016), the terms $A_{n,m,x}(x)$ and $C_{n,2,m}(x)$ are bounded for fixed $x$ and $m$, when $n \to \infty$ where for the operators $J_a$, we have

$$A_{n,m,x}(x) = J_n\left[1+\left(x + \frac{|t-x|}{2}\right)^m\right];x = 1 + 2\sum_{k=0}^{m}\frac{(m)}{k!}x^kJ_a(|t-x|^{m-k},x)\frac{1}{2^{m-k}} + \sum_{k=0}^{m}\frac{(2m)}{k!}x^kJ_a(|t-x|^{2m-k},x)\frac{1}{2^{2m-k}}$$

and $C_{n,2,m}(x) = 1 + \frac{1}{J_a(|y-x|^3,x)}\sum_{k=0}^{m}\frac{(m)}{k!}x^kJ_a(|y-x|^{k+3},x)\frac{1}{2^k}.$ (8.2)

Now we apply Theorem 2.2 and Theorem 2.3 of Tachev and Gupta (2016) for our operator and along the lines of Tachev and Gupta (2016), we obtain the following results:

**Theorem 8.5.1.** If $f \in C^2[0,\infty) \cap P$ and $f'' \in W_\phi[0,\infty)$, then we have for $x \in (0,\infty)$ that

$$\left|J_a(f;x) - f(x) - \frac{x(x+3)}{2(n-1)}f''(x)\right| \leq \frac{1}{2}\left[\frac{x(x+3)}{n-1} + \sqrt{2A_{n,m,x}}\right] \times \omega_\phi\left(f'';\sqrt{M_{n,6}(x)}\right),$$

where $A_{n,m,x}(x)$ is given by Eqn. (8.2) and $M_{n,6}(x)$ is given as in Remark 8.2.1.

The quantitative variant of Voronovskaja’s theorem for our operator is as follows:

**Theorem 8.5.2.** If $f \in C^2[0,\infty) \cap P$ and $f'' \in W_\phi[0,\infty)$, then we have for $x \in (0,\infty)$ that

$$\left|(n-1)\left[J_a(f;x) - f(x) - \frac{x(x+3)}{n-1}f''(x)\right]\right|$$

$$\leq \frac{1}{2}\left|x(x+3) + \sqrt{2}x(x+3)C_{n,2,m}(x)\right| \times \omega_\phi\left(f'';\sqrt{M_{n,6}(x)}\right),$$

Where $M_{n,2}(x)$ and $M_{n,d}(x)$, are given as in Remark 8.2.1 and $C_{n,2,m}(x)$ is given by Eqn. (8.2).
It is assumed for the operators $J_n$ that

$$\frac{J_n(|y-x|^k,x)}{J_n(|y-x|^\beta,x)}, 4 \leq k \leq m$$

is a bounded ratio for fixed $x$ and $m$, when $n \to \infty$.

8.6. CONCLUSION

In this Chapter, we studied the approximation properties of genuine Lupas-Beta operators of integral type $J_n(f,x)$. We found moment estimates required for analysis up to order six by using factorial polynomial and an elementary hypergeometric function. We established a quantitative asymptotic formula and some direct estimate in terms of Ditzian-Totik modulus of continuity. At the end, we mentioned results on the weighted modulus of continuity due to Păltănea for the operators $J_n(f,x)$. 