# Chapter 3

# Derivation of Reynolds Equation

## 3.1 Mathematical Modelling of a Bearing System

The mathematical modeling of the bearing system dates back to the research developments in the field of fluid dynamics of real fluids which started in nineteenth century. In fact, Hydrodynamic film lubrication was effectively used before it was scientifically understood. The process of lubrication is basically, a part of overall phenomena of hydrodynamics whose scientific analysis was initiated during nineteenth century. Adams (1853) was first to develop and patent several good designs for railway axle bearing in 1847. The understanding of hydrodynamic lubrication began with the classical experiments of Tower (1884, 1885) in connection with the investigation of friction of the railway partial journal bearing when he measured the lubricant pressure in the bearing. This work was modified by Reynolds (1886). He applied hydrodynamic laws to the bearing problem and explained Tower's results satisfactorily. He derived and employed an equation for the analysis of fluid film lubrication which has by now become a basic governing equation and is named after him as Reynolds equation.

Reynolds combined Navier-Stokes equations with continuity equation to generate a second order differential equation for lubricant pressure. Subsequently, it was realized that the Reynolds equation is valid only over a much narrower field than is generally supposed to be. The so called conventional Reynolds equation contains viscosity, density and film thickness as parameters. These parameters determine and depend on the temperature and the pressure fields and on the elastic behavior of the bearing surfaces. Besides these, sometimes surface roughness, porosity and other increased severity of bearing operating conditions etc. may demand the need to generalize Reynolds equation accordingly to account for these effects. Likewise, consistent with these effects and the requirement of the particular bearing problems, it may become necessary to relax few of the assumptions used for derivation of the Reynolds equation. Thus, the study of hydrodynamic lubrication is from a mathematical point of view is in fact, the study of a particular form of Navier-Stokes equations compatible with the system. Since the Reynolds time, researches in the field of lubrication have made significant progress with the rapid advancement of machines, manufacturing process and materials in which lubrication plays an important role; the study of lubrication has gained considerable importance and has become, from analytical point of view, an independent branch of fluid mechanics.

Mathematical modeling of a bearing system consists of various conservation laws of fluid dynamics such as conservation of mass, momentum, energy and equation describing various aspects characterizing the bearing problem such as constitutive equation of lubricant, viscosity dependence on pressure, temperature, equation of state, elastic deformations, surface roughness etc.

## 3.2 Equation of State

Phenomenological consideration requires specification of the state of fluid which is given by an equation which is called equation of state. For an incompressible fluid, it is given by

$$\rho = \text{constant}$$

while for a perfect gas for isothermal variations in pressure it is given by Boyle-Mariotte law as

$$P = \rho R T \tag{3.1}$$

where $R$ is the universal gas constant.

For constant compressibility fluids under isothermal conditions, equation of state is

$$\rho = \rho_0 \exp\left[C\left(P - P_0\right)\right] \tag{3.2}$$

where $\rho_0$ is the value of $\rho$ at the reference atmospheric pressure $P_0$ and $C$ is the compressibility. This particular equation of state applies rather well to most liquids.

## 3.3 Constitutive Equation of Lubricant

The mathematical equation governing the viscous contribution to the stress tensor with the rate of deformation tensor is called constitutive equation applicable to the description of rheological behavior of the lubricant. The constitutive equations are of three types namely integral type, rate type, and differential type. Many lubricating fluids are generally, Newtonian and in such cases, shearing stress is directly proportional to rate of strain tensor, constant of proportionality being the dynamic viscosity of the lubricant. The lubricant being Newtonian in character greatly simplifies the mathematical analysis. The lubricants which exhibit a relationship other than that exists for a Newtonian lubricant are generally called non-Newtonian lubricant.

Significant lubricating fluid properties are viscosity, density, specific heat and thermal conductivity. Among these fluid properties, viscosity plays a more prominent role. Viscosity varies with temperature as well as pressure and this variation is important in lubrication mechanics. There is a general rule that more viscous lubricant is more susceptible to change. In general, viscosity increases with pressure and decreases with temperature for most liquid, lubricants.

The viscosity of heavily loaded lubricating film is generally treated as a function of both pressure and temperature. Barus (1893) approximated the viscosity for limited ranges by

$$\eta = \mu_0 \exp\left[a\left(P - P_0\right) + b\left(T - T_0\right)\right] \tag{3.3}$$

where a and b are called pressure and temperature viscosity coefficient and the subscript ‘0’ refers to atmospheric conditions. Over reasonably large ranges of temperature and pressure, the linear relation

$$\eta = \mu_0 \exp\left[1 + a\left(P - P_0\right) + b\left(T - T_0\right)\right] \tag{3.4}$$

turns out to be useful.

## 3.4 Continuity Equation

For a compressible fluid, the continuity equation is

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \vec{q}) = 0 \tag{3.5}$$

where $\vec{q}$ is the velocity vector of the flowing fluid and $\rho$ is the density. If the flow is steady $\frac{\partial \rho}{\partial t} = 0$ and hence the continuity equation turns out to be

$$\nabla.(\rho \vec{q}) = 0 \tag{3.6}$$

The equation of continuity for homogeneous, incompressible fluid takes the form

$$\nabla.\vec{q} = 0 \tag{3.7}$$

A comparison of equations (3.5) and (3.7) suggests that the density of the fluid does not appear in the continuity equation for incompressible fluids, whereas it does appear in the corresponding equation for compressible fluids. Thus, the continuity equation for incompressible fluids is a purely kinematical equation whereas for compressible fluids, it is a dynamical one.

## 3.5 The Equations of Motion

Principle of conservation of momentum when applied to fluid, contained in a control volume states that forces acting on the fluid in the control volume equal the rate of outflow of momentum from the control volume through the closed surface enclosing it. The mathematical equation expressing this condition for Newtonian, isoviscous, laminar, continuum and compressible fluid flow for which body forces such as gravitational forces or electromagnetic forces etc. are considered negligible, is given by.

$$\rho \frac{\partial \vec{q}}{\partial t} + \rho(\vec{q}.\nabla)\vec{q} = -\nabla p + (\lambda + \eta)\nabla(\nabla.\vec{q}) + \eta \nabla^2 \vec{q} \tag{3.8}$$

where $\eta$ is called the coefficient of shear viscosity of the fluid and $\lambda$ is called coefficient of bulk viscosity. It is often assumed that they are related by

$$3\lambda + 2\eta = 0$$

The equation (3.8), were first obtained by Navier in 1821 and later independently by Stokes in 1845. Hence these are known as Navier-Stokes equations. The first term on the left hand side of equation (3.8) is temporal acceleration term while the second is convective inertia term. The first term on right hand side is due to pressure and the other terms are consequences of viscous forces. If, however, the fluid is incompressible, as is the case with most liquid lubricants, then

$$\nabla.\vec{q} = 0$$

and equation (3.8) gets simplified to

$$\rho \frac{\partial \vec{q}}{\partial t} + \rho\left(\vec{q}.\nabla\right)\vec{q} = -\nabla p + \eta \nabla^2 \vec{q} \tag{3.9}$$

When a large external electromagnetic field through the electrically conducting lubricant is applied, it gives rise to induced circulating currents, which in turn, interacts with the magnetic field and creates a body force called Lorentz force. This extra electromagnetic pressurization pumps the fluid between the bearing surfaces. In such a case Navier-Stokes equations for an incompressible isoviscous liquid gets modified as

$$\rho \frac{\partial \vec{q}}{\partial t} + \rho\left(\vec{q}.\nabla\right)\vec{q} = -\nabla p + \eta \nabla^2 \vec{q} + \vec{J} \times \vec{B} \tag{3.10}$$

where $\vec{J}$ is the electric current density and $\vec{B}$ is the magnetic induction vector. In this case, Maxwell's equations and Ohm's law are taken into account. These are,

$$\nabla \times \vec{B} = \mu_0 \vec{J}, \quad \nabla.\vec{B} = 0$$

$$\vec{J} = \sigma\left[\vec{E} + \vec{q} \times \vec{B}\right] \tag{3.11}$$

$$\nabla \times \vec{E} = 0, \quad \nabla.\vec{E} = 0$$

where $\vec{E}$ is the electric field intensity vector, $\sigma$ is the electrical conductivity and $\mu_0$ is the magnetic permeability of the lubricant.

## 3.6 Surface Roughness

In most of the theoretical studies of fluid film lubrication, it has more or less explicitly been assumed that the bearing surfaces can be represented by smooth mathematical planes. It has, however long been recognized that this might be an unrealistic assumption, particularly, in bearing working with small film thickness that is in boundary and mixed lubrication regimes. Several mathematical methods, such as postulating a sinusoidal variation in film thickness have been introduced in order to seek a more realistic representation of rubbing surfaces. However, this method is perhaps, more appropriate in an analysis of the influence of waviness rather than roughness.

Earlier attempts for mathematical modeling of the rough bearing surfaces used the postulation by a saw-tooth curve (Davies (1963)) or represented by waviness approximated by a Fourier type series (Burton (1963)). Tzeng and Saibel (1967) recognized the random character of roughness and employed a method of random analysis and assumed the one dimensional film thickness to be of the form

$$h(x,\varepsilon) = h(x) + \bar{h}(x) \tag{3.12}$$

where $\bar{h}(x)$ constitutes the stochastic variation from the smooth film thickness $h(x)$. Thickness $\bar{h}(x)$ is regarded as a random variable whose probability density function is either a Gaussian normal probability distribution function or a beta-distribution function given by

$$f\left(h_s\right) = \begin{cases} \dfrac{35}{32c^7}\left(c^2 - h_s{}^2\right)^3, & -c \le h_s \le c \\ 0, & elsewhere \end{cases} \tag{3.13}$$

where c is the maximum deviation from the mean film thickness. This distribution function is used to average out the physical quantities in the Reynolds equation with respect to film thickness. In addition to this mathematical form, many other mathematical approaches have also been made in a number of investigations.

## 3.7 Basic Assumptions of Hydrodynamic Lubrication

For the analysis that follows to derive the modified Reynolds equation, usually following assumptions are made:

1. The lubricant is considered to be incompressible, non-conducting and non-magnetic with constant density and viscosity, unless otherwise stated. Most of lubricating fluids usually satisfy this condition.

2 Flow of the lubricant is laminar, unless otherwise stated. A moderate velocity combined with a high kinematic viscosity gives rise to a low Reynolds number, at which flow essentially remains laminar.

3 Body forces are neglected, i.e. there are no external fields of force acting on the fluid. While magnetic and electrical forces are not present in the flow of non-conducting lubricants, forces due to gravitational attraction are always present. However, these forces are small enough as compared to the viscous force involved. Thus, they are usually neglected in lubrication mechanics without causing any significant error.

4 Flow is considered steady, unless otherwise stated, i.e. velocities and fluid properties do not vary with time. Temporal acceleration due to velocity fluctuations are small enough in comparison with lubricant inertia, hence may usually be ignored.

5 Boundary layer is assumed to be fully developed throughout the lubricating region so that entrance effects at the leading edge and the film discontinuity at the trailing edge from which vortices may be shed, are neglected.

6 A fundamental assumption of hydrodynamic lubrication is that the thickness of the fluid is considered very small in comparison with the dimensions of the bearings. As a consequences of this assumptions:

- The curvature of the film may be neglected, so that bearing surfaces may be considered locally straight in direction.
- Fluid inertia may be neglected when compared with viscous forces.
- Since lubricant velocity along the transverse direction to the film is small, variation of pressure may also be neglected in this direction.
- Velocity gradients across the film predominate as compared to those in the plane of the film.

7 The fluid behaves as a continuum which implies that pressures are high enough so that the mean free path of the molecule of the fluid is much smaller than the effective pore diameter or any other dimension. No slip boundary condition is applicable at the bearing surfaces.

8 Lubricant film is assumed to be isoviscous.

9 Temperature changes due to the lubricant are neglected.

10 The bearing surfaces are assumed to be perfectly rigid so that elastic deformation of the bearing surfaces may be neglected.

11 In case of bearing working with magnetic fluids, the lubricant is assumed to be free of charged particles.

12 When bearings work under the influence of electromagnetic fields, it is assumed that the forces due to induction are small enough to be neglected.

## 3.8 Modified Reynolds equation for porous bearings in Cartesian coordinates

The differential equation which is developed by making use of the assumptions of hydrodynamic lubrication in equations of motion and continuity equation and combining them into a single equation governing lubricant pressure is called Reynolds equation. The Reynolds equation when derived for more general situations like porous bearings or hydro magnetic bearings or bearings working with non-Newtonian or magnetic lubricant, etc. is called generalized Reynolds equation or modified Reynolds equation.

At this point one recalls that the differential equation originally derived by Reynolds (1886) is restricted to incompressible fluids. This however, is an unnecessary restriction, for the equation can be formulated broadly enough to include effects of compressibility and dynamic loading.

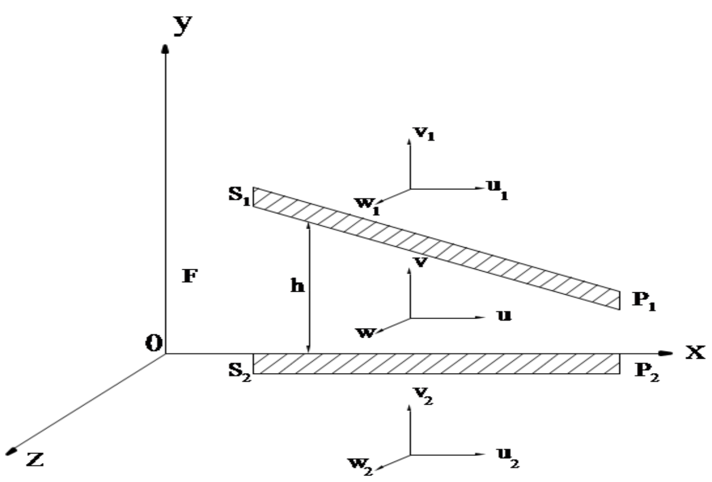


**Fig. 3.1 Configuration of Slider Bearing**

Consider that the upper surface of the bearing surfaces is $S_1$ and the lower surface is $S_2$ which are in relative motion with uniform velocities $(U_1, V_1, W_1)$ and $(U_2, V_2, W_2)$ respectively. The surface $S_1$ and $S_2$ enclose the lubricant film. The lubricant velocities in the film region $F, S_1$ and $S_2$ are $(u, v, w)$, $(u_1, v_1, w_1)$ and $(u_2, v_2, w_2)$ respectively. Lubricant pressure, in $F, S_1$ and $S_2$ are $p$, $P_1$ and $P_2$ respectively. Film thickness $h$ is assumed to be a function of $x$.

1 There is no variation of pressure across the fluid film, results in $\frac{dp}{dz} = 0$.

2 The flow is laminar; no vertex of flow and no turbulence occur anywhere in the film.

3 No external forces act on the film. Thus, $X = Y = Z = 0$

4 Fluid inertia is small compared to the viscous shear. This consideration leads to,

$$\frac{du}{dt} = \frac{dv}{dt} = \frac{dw}{dt} = 0$$

5 Usually, no slip condition is taken in to account at the bearing surfaces.

6 The height of the fluid film z is very small compared to the span and length $x, y$. This allows us to ignore the curvature of the fluid film, such as in the case of journal bearings, and replace rotational by translational velocities.

7 Compared with the two velocity gradients $\frac{du}{dz}$ and $\frac{dv}{dz}$ all other velocity gradients are considered negligible. We can thus omit all derivatives with the exception of $\frac{d^2u}{dz^2}$ and $\frac{d^2v}{dz^2}$.

8 The porous region is homogeneous and isotropic.

9 The lubricant is Newtonian with constant density and viscosity.

10 The flow in the porous region is governed by Darcy's law:

$$\overline{Q}_1 = -\frac{\overline{k}}{\eta} \nabla P_1$$

11 Where $\overline{Q}_1, \overline{k}, \eta$ and $P_1$ are respectively velocity, permeability, viscosity and pressure of the fluid in the porous region.

12 Bearing is press-fitted in a solid housing.

13 Pressure and normal velocity components are continuous at the interface.

The equation of motion under the assumptions stated above, take the form

$$-\frac{\partial p}{\partial x}+\eta\frac{\partial^2 u}{\partial z^2}=0 \tag{3.14}$$

$$-\frac{\partial p}{\partial y}+\eta\frac{\partial^2 v}{\partial z^2}=0 \tag{3.15}$$

and

$$-\frac{\partial p}{\partial z}=0 \tag{3.16}$$

From equations (3.14) and (3.16) one finds that

$$\eta\frac{\partial^2 u}{\partial z^2}=\frac{\partial p}{\partial x} \tag{3.17}$$

$$\eta\frac{\partial^2 v}{\partial z^2}=\frac{\partial p}{\partial y} \tag{3.18}$$

The no-slip boundary conditions are

$$u=U_1, v=V_1, w=W_1 \text{ at } z=0$$

$$u=U_2, v=V_2, w=W_2 \text{ at } z=h$$

By integrating equation (3.17) twice with the above boundary conditions one finds that

$$\frac{\partial u}{\partial z}=\frac{1}{\eta}\frac{\partial p}{\partial x}z+A$$

resulting in

$$u=\frac{1}{\eta}\frac{\partial p}{\partial x}\frac{z^2}{2}+Az+B$$

Similarly, one concludes that

$$v = \frac{1}{\eta}\frac{\partial p}{\partial y}\frac{z^2}{2} + A_1 z + B_1 \qquad (3.19)$$

We now make use of the continuity equation (3.6) with no source or sinks present and with the state of the lubricant independent of time, the continuity equation emerges to be

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0 \qquad (3.20)$$

The equation for homogeneous incompressible fluid takes the form

$$\nabla.\vec{q} = 0$$

Equivalently, one arrives at

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad (3.21)$$

Solutions of equations (3.19) with related boundary conditions are

$$u = \frac{1}{2\eta}\frac{\partial p}{\partial x}\left(z^2 - hz\right) + \left(U_2 - U_1\right)\frac{z}{h} + U_1,$$

$$v = \frac{1}{2\eta}\frac{\partial p}{\partial y}\left(z^2 - hz\right) + \left(V_2 - V_1\right)\frac{z}{h} + V_1 \qquad (3.22)$$

Substituting values of $u$ and $v$ in equation (3.21), one can see that

$$\frac{\partial w}{\partial z} = -\frac{\partial}{\partial x}\left[\frac{1}{2\eta}\frac{\partial p}{\partial x}\left(z^2 - hz\right) + \left(U_2 - U_1\right)\frac{z}{h} + U_1\right]$$

$$-\frac{\partial}{\partial y}\left[\frac{1}{2\eta}\frac{\partial p}{\partial y}\left(z^2 - hz\right) + \left(V_2 - V_1\right)\frac{z}{h} + V_1\right]$$

By integrating across the film thickness that is from $z = 0$ to $z = h$, one obtains

$$W_2 - W_1 = \frac{\partial}{\partial x}\left[\frac{h^3}{12\eta}\frac{dp}{dx}\right] - \frac{\partial}{\partial x}\left[\left(U_2 + U_1\right)\frac{h}{2}\right]$$

$$+\frac{\partial}{\partial y}\left[\frac{h^3}{12\eta}\frac{\partial p}{\partial y}\right] - \frac{\partial}{\partial y}\left[\left(V_2 + V_1\right)\frac{h}{2}\right]$$

which can be written by rearranging the terms as

$$\frac{\partial}{\partial x}\left[h^3\frac{\partial p}{\partial x}\right] + \frac{\partial}{\partial y}\left[h^3\frac{\partial p}{\partial y}\right] = 12\eta\left[\frac{\partial}{\partial x}\left[\left(U_1 + U_2\right)\frac{h}{2}\right] + \frac{\partial}{\partial y}\left[\left(V_1 + V_2\right)\frac{h}{2}\right] + \left(W_2 - W_1\right)\right] \qquad (3.23)$$

This equation is known as Generalized Reynolds equation for an incompressible fluid.

In most applications we consider the upper surface as non porous and moving with a uniform velocity $U_2 = U$ in the x-direction together with a normal velocity $W_2$ and the lower surface is stationary and has a porous facing of thickness $H^*$. Due to continuity of velocities at the interfaces,

$$W_1 = \frac{\bar{k}}{\eta}\left(\frac{\partial P_1}{\partial z}\right)_{z=0} \text{ and } W_2 = \frac{\bar{k}}{\eta}\left(\frac{\partial P_2}{\partial z}\right)_{z=h}.$$

the modified Reynolds equation for a porous bearings take the form

$$\frac{\partial}{\partial x}\left[h^3\frac{\partial p}{\partial x}\right]+\frac{\partial}{\partial y}\left[h^3\frac{\partial p}{\partial y}\right] = 12\eta\left[\frac{\partial}{\partial x}\left[(U_1+U_2)\frac{h}{2}\right]+\frac{\partial}{\partial y}\left[(V_1+V_2)\frac{h}{2}\right]\right]$$

$$-12\bar{k}\left[\left(\frac{\partial P_2}{\partial z}\right)_{z=h}-\left(\frac{\partial P_1}{\partial z}\right)_{z=0}\right] \tag{3.24}$$

In most of the applications of bearing systems we consider one of the surfaces as non-porous and moving with a uniform velocity U in the x-direction together a normal velocity $W_h$. Particularly, let us consider that lower surface is stationary and has a porous facing of thickness $H^*$. Therefore,

$$U_1 = 0, U_2 = U, V_1 = V_2 = 0,\ W_1 = 0, P_1 = P,\ W_2 = W_h \text{ and } \left(\frac{\partial P_2}{\partial z}\right)_{z=h} = 0$$

Accordingly equation (3.24) reduces to

$$\frac{\partial}{\partial x}\left[h^3\frac{\partial p}{\partial x}\right]+\frac{\partial}{\partial y}\left[h^3\frac{\partial p}{\partial y}\right] = 6\eta U\frac{\partial h}{\partial x}+12\eta W_h+12\bar{k}\left(\frac{\partial P}{\partial z}\right)_{z=0} \tag{3.25}$$

where the pressure $P$ in the porous region satisfies the Laplace equation

$$\frac{\partial^2 P}{\partial x^2}+\frac{\partial^2 P}{\partial y^2}+\frac{\partial^2 P}{\partial z^2} = 0 \tag{3.26}$$

Using the Morgan and Cameron (1957) approximation that when $H^*$ is small, the pressure in the porous region can be replaced by the average pressure with respect to the bearing wall thickness and that was extensively used by Prakash and Vij (1973 a). It is uncoupled by substituting

$$\left(\frac{\partial P}{\partial z}\right)_{z=0} = -H^* \left(\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2}\right) \tag{3.27}$$

Thus, the modified Reynolds equation comes out to be

$$\frac{\partial}{\partial x}\left[\left(h^3 + 12\bar{k}H^*\right)\frac{\partial p}{\partial x}\right] + \frac{\partial}{\partial y}\left[\left(h^3 + 12\bar{k}H^*\right)\frac{\partial p}{\partial y}\right] = 6\eta U\frac{\partial h}{\partial x} + 12\eta W_h \tag{3.28}$$

Hence, the problem of finding the film pressure is reduced to the solution of equation (3.28) with appropriate boundary conditions.

## 3.9 Derivation of equations for Neuringer-Rosensweig model

In 1964, Neuringer and Rosensweig investigated a simple flow model to describe the steady flow of magnetic fluids in the presence of slowly changing external magnetic fields. The model consists of the following equations:

$$\rho(\bar{q}\nabla)\bar{q} = -\nabla p + \eta\nabla^2\bar{q} + \mu_0(\bar{M}\nabla)\bar{H} \tag{3.29}$$

$$\nabla\bar{q} = 0 \tag{3.30}$$

$$\nabla\times\bar{H} = 0 \tag{3.31}$$

$$\bar{M} = \bar{\mu}\,\bar{H} \tag{3.32}$$

$$\nabla\left(\bar{H} + \bar{M}\right) = 0 \tag{3.33}$$

Where $\rho$ represent the fluid density, $\bar{q} = (u, v, w)$ was the fluid velocity in the film region, $p$ was the film pressure, $\eta$ represented the fluid viscosity, $\mu_0$ denoted the permeability of free space, $\bar{M}$ was the magnetization vector, $\bar{H}$ denoted the external magnetic field and $\bar{\mu}$ was the magnetic susceptibility of the magnetic particles.

Using above equations (3.31) and (3.32), equation (3.29) assumes the form

$$\rho(\bar{q}\nabla)\bar{q} = -\nabla\left(p - \frac{\mu_0\bar{\mu}}{2}H^2\right) + \eta\nabla^2\bar{q}$$

This suggested that an extra pressure $\frac{\mu_0\bar{\mu}}{2}H^2$ is introduces in to the Navier-Stokes equations when a magnetic fluid is used as a lubricant. Thus, the modified Reynolds equation in this case is derived like equation (3.28) as

$$\frac{\partial}{\partial x}\left(\left(h^3+12\bar{k}H^*\right)\frac{\partial}{\partial x}\left(p-\frac{\mu_0\bar{\mu}}{2}H^2\right)\right)+\frac{\partial}{\partial y}\left(\left(h^3+12\bar{k}H^*\right)\frac{\partial}{\partial y}\left(p-\frac{\mu_0\bar{\mu}}{2}H^2\right)\right)$$

$$=6\eta U\frac{\partial h}{\partial x}+12\eta W_h \qquad (3.34)$$

The modified equation for solid surface is obtained by setting $H^* = 0$ in above equation.

## 3.10 Derivation of equations for Shliomis model

Shliomis (1972) established that magnetic particles of a magnetic fluid can relax in two ways when the applied magnetic field changes. The first is by the rotation of magnetic particles in the fluid and the second by rotation of the magnetic moment with in the particles. Brownian relaxation time parameter $\tau_B$ gives particle rotation while the relaxation time parameter $\tau_S$ describes the intrinsic rotational process. Assuming steady flow, neglecting inertial and second derivatives of $\bar{S}$, the equations governing the flow turns out to be,

$$-\nabla p+\eta\nabla^2\bar{q}+\mu_0\left(\bar{M}\nabla\right)\bar{H}+\frac{1}{2\tau_S}\nabla\times\left(\bar{S}-I\bar{\Omega}\right)=0 \qquad (3.35)$$

$$\bar{S}=I\bar{\Omega}+\mu_0\tau_S\left(\bar{M}\times\bar{H}\right) \qquad (3.36)$$

$$\bar{M}=M_0\frac{\bar{H}}{H}+\frac{\tau_B}{I}\left(\bar{S}\times\bar{M}\right) \qquad (3.37)$$

where $\bar{S}$ is the internal angular momentum, $I$ is the sum of moments of inertia of the particles per unit volume, $\bar{\Omega}=\frac{1}{2}\nabla\times\bar{q}$ together with equations (3.30), (3.31) and (3.33) (Bhat (2003)).

By making use of equation (3.36), in equation (3.35) and (3.37), one finds that,

$$-\nabla p+\eta\nabla^2\bar{q}+\mu_0(\bar{M}\nabla)\bar{H}+\frac{1}{2}\mu_0\nabla\times(\bar{M}\times\bar{H})=0 \qquad (3.38)$$

and

$$\bar{M}=M_0\frac{\bar{H}}{H}+\tau_B\left(\bar{\Omega}\times\bar{M}\right) \qquad (3.39)$$

Neglecting $\tau_B\ \tau_S$ terms and substitution of $\bar{M}$ in above equation, leads to

$$-\nabla p+\left(\eta+\frac{\mu_0}{4}\tau_B\bar{M}\,\bar{H}\right)\nabla^2\bar{q}+\mu_0(\bar{M}\nabla)\bar{H}$$

$$+\frac{1}{2}\mu_0\tau_B\left[\nabla\left(\bar{\Omega}\bar{H}\right)\times\bar{M}+\left(\bar{\Omega}\bar{H}\right)\nabla\times\bar{M}-\nabla(\bar{M}\bar{H})\times\bar{\Omega}\right]=0 \qquad (3.40)$$

From equation (3.39), it is easily observed that an initial approximation to $\bar{M}$ can be

$$\bar{M}=M_0\frac{\bar{H}}{H}$$

Substituting the value of $\bar{M}$ on the right side of equation (3.39), a second approximation to $\bar{M}$ is found to be

$$\bar{M}=M_0\frac{\bar{H}}{H}+\frac{M_0}{H}\tau_B\left(\bar{\Omega}\times\bar{H}\right)$$

Again, substituting this value of $\bar{M}$ on the right side of equation (3.39), third approximation to $\bar{M}$ is obtained as

$$\bar{M}\bar{H}=M_0H+\frac{M_0}{H}\tau_B^2\left\{\left(\bar{\Omega}\bar{H}\right)^2-\Omega^2H^2\right\} \qquad (3.41)$$

Therefore, Shliomis model of magnetic fluid flow is governed by the equations (3.30), (3.31), (3.33), (3.40) and (3.41).

Then proceeding along the analysis adopted in Bhat [2003]. Reynolds type equations for Shliomis model for a one dimensional flow for impermeable slider bearing with the slider moving with a uniform velocity U in the x-direction, is obtained as

$$\frac{d}{dx}\left(h^3\frac{dp}{dx}\right)=12\eta_a\dot{h}_0+6\eta_aU\frac{dh}{dx}-\frac{3N\tau_B{}^3U^2}{16\eta_a}\frac{d}{dx}\left(h\frac{dp}{dx}\right)-\frac{3N\tau_B{}^3}{320\eta_a{}^3}\frac{d}{dx}\left(h^5\left(\frac{dp}{dx}\right)^3\right) \qquad (3.42)$$

## 3.11 Derivation of equations for Jenkins model

In 1972, the flow model of a Ferro-fluid was discussed by Jenkins. In view of the Maugin's modifications, equations of the model for steady flow are (Jenkins (1972) and Ram and Verma (1999))

$$\rho(\bar{q}\nabla)\bar{q}=-\nabla p+\eta\nabla^2\bar{q}+\mu_0\left(\bar{M}\nabla\right)\bar{H}+\frac{\rho A^2}{2}\nabla\times\left[\frac{\bar{M}}{M}\times\left\{(\nabla\times\bar{q})\times\bar{M}\right\}\right] \qquad (3.43)$$

together with equations (3.29)-(3.33) and *A* being a material constant parameter.

From the above equation it is noticed that Jenkins model is a generalization of Neuringer-Rosensweig model with an additional term

$$\frac{\rho A^2}{2}\nabla\times\left[\frac{\bar{M}}{M}\times\left\{(\nabla\times\bar{q})\times\bar{M}\right\}\right]=\frac{\rho A^2\bar{\mu}}{2}\nabla\times\left[\frac{\bar{H}}{H}\times\left\{(\nabla\times\bar{q})\times\bar{H}\right\}\right] \tag{3.44}$$

which modifies the velocity of the fluid. Neuringer- Rosensweig modifies the pressure while Jenkins model modifies both the pressure and velocity of the Magnetic fluid.

Then proceeding along the analysis discussed in Bhat [2003]. Generalized Reynolds type equations for Jenkins model for a one dimensional flow for impermeable slider bearing with the slider moving with a uniform velocity U in the x-direction, is obtained as

$$\frac{d}{dx}\left[\frac{h^3}{1-\frac{\rho A^2\bar{\mu}H}{2\eta}}\frac{d}{dx}\left(p-\frac{\mu_0\bar{\mu}}{2}H^2\right)\right]=6\eta U\frac{dh}{dx}+12\eta W_h \tag{3.45}$$

Incorporating the roughness term and the term containing porosity in equation (3.45) one can get,

$$\frac{d}{dx}\left[\frac{g(h)}{1-\frac{\rho A^2\bar{\mu}H}{2\eta}}\frac{d}{dx}\left(p-\frac{\mu_0\bar{\mu}}{2}H^2\right)\right]=6\eta U\frac{dh}{dx}+12\eta W_h \tag{3.46}$$

Where,

$$g(h)=h^3+3h^2\alpha+3\left(\alpha^2+\sigma^2\right)h+3\sigma^2\alpha+\alpha^3+\varepsilon+12\phi H$$

Equation (3.46) express the generalized modified Reynolds type equation for magnetic fluid based transversely rough bearing system.