Chapter 4

The $b$-chromatic number of some wheel related graphs
Chapter 4. The $b$-chromatic number of some wheel related graphs

4.1 Introduction

A brief account of $b$-coloring and its related concepts are given in previous chapter 3 while the present chapter is aimed to report the investigations related to $b$-coloring of some wheel related graphs.

4.2 $b$-coloring of helm, closed helm and flower graph

Lemma 4.2.1. For the helm $H_n$

\[ \chi(H_n) = \begin{cases} 
3, & n \text{ even} \\
4, & n \text{ odd}.
\end{cases} \]

Proof. Let $\{e_1, e_2, \ldots, e_n\}$ be the spoke edges of $H_n$ and $\{e_{n+1}, e_{n+2}, \ldots, e_{2n}\}$ be the rim edges of the cycle in $H_n$ while $\{e_{2n+1}, e_{2n+2}, \ldots, e_{3n}\}$ be the pendant edges of $H_n$. Moreover $\{u_1, u_2, \ldots, u_n\}$ be the pendant vertices of $H_n$ and $\{v_1, v_2, \ldots, v_n\}$ be the vertices of degree 4. Denote the apex of $H_n$ as $v$. Also $|V(H_n)| = 2n + 1$ and $|E(H_n)| = 3n$. To prove the result we consider the following cases.

Case 1: $n$ even

In this case $H_n$ contains a cycle $C_3$. Then by Proposition 3.3.3, $\chi(H_n) \geq 3$. If we assign a proper coloring as $f(v) = 3$, $f(v_{2k-1}) = 1$, $f(v_{2k}) = 2$, $f(u_{2k-1}) = 2$, $f(u_{2k}) = 1; k \in \mathbb{N}$ then $\chi(H_n) = 3$.

Case 2: $n$ odd

In this case $H_n$ contains a cycle $C_3$. Then by Proposition 3.3.3, $\chi(H_n) \geq 3$. As $v$ is one of the vertex of $C_3$ and we have already used 3-colors for proper coloring. Therefore the fourth color is needed for $v$. Hence $\chi(H_n) = 4$.

Thus

\[ \chi(H_n) = \begin{cases} 
3, & n \text{ even} \\
4, & n \text{ odd}.
\end{cases} \]
Theorem 4.2.2.

\[
\varphi(H_n) = \begin{cases} 
4, & n = 3 \\
5, & n = 4 \\
4, & n = 5 \\
5, & n = 6 \\
5, & n \geq 7.
\end{cases}
\]

Proof. We continue with the terminology and notations used in Lemma 4.2.1 and consider the following cases.

Case 1: For \( n = 3 \)

In this case, \(|V(H_3)| = 7\) and \(|E(H_3)| = 9\). Also \(m(H_3) = 4\). Then by Proposition 3.3.4, \(\varphi(H_3) \leq 4\). Further more the graph contains \(K_3\), then according to Proposition 3.3.5, \(\varphi(H_3) \geq 3\). If possible \(H_3\) has b-coloring using four colors with \(c = \{1, 2, 3, 4\}\). To assign the proper coloring define the color function \(f : V(H_3) \to \{1, 2, 3, 4\}\) as \(f(v) = 4, f(v_1) = 1, f(v_2) = 2, f(v_3) = 3, f(u_1) = 2, f(u_2) = 1, f(u_3) = 2\). This proper coloring gives the color dominating vertices as \(cdv(1) = v_1, cdv(2) = v_2, cdv(3) = v_3, cdv(4) = v\). Thus \(\varphi(H_3) = 4\).

Case 2: For \( n = 4 \)

In this case, \(|V(H_4)| = 9\), \(|E(H_4)| = 12\) and \(m(H_4) = 5\). As \(H_4\) contains exactly 5 vertices of degree \(m(H_4) - 1\) then \(H_4\) is a tight graph. Also \(\chi(H_4^+) = m(H_4) = 5\). Then by Proposition 3.3.6, \(\varphi(H_4) = 5\).

For b-coloring consider the color class \(c = \{1, 2, 3, 4, 5\}\) and to assign the proper coloring we define the color function \(f : V(H_4) \to \{1, 2, 3, 4, 5\}\) as \(f(v) = 5, f(v_1) = 1, f(v_2) = 2, f(v_3) = 3, f(v_4) = 4, f(u_1) = 3, f(u_2) = 4, f(u_3) = 1, f(u_4) = 2\). This proper coloring gives the color dominating vertices as \(cdv(1) = v_1, cdv(2) = v_2, cdv(3) = v_3, cdv(4) = v_4, cdv(5) = v\). Thus \(\varphi(H_4) = 5\).

Case 3: For \( n = 5 \)

In this case, \(|V(H_5)| = 11\) and \(|E(H_5)| = 15\). Also \(m(H_5) = 5\). Then by Proposition 3.3.4, \(\varphi(H_5) \leq 5\). Suppose that \(H_5\) does have a b-chromatic 5-coloring. Now consider the color class \(c = \{1, 2, 3, 4, 5\}\) and to assign the proper coloring we define the color
function as \( f : V(H_5) \to \{1, 2, 3, 4, 5\} \) as \( f(v) = 5, f(v_1) = 4, f(v_2) = 1, f(v_3) = 2, f(v_4) = 3, f(u_1) = 3, f(u_2) = 4, f(u_3) = 4, f(u_4) = 2, f(u_5) = 2 \) which in turn forces us to assign \( f(v_5) = 1 \). This proper coloring gives the color dominating vertices for color classes 1, 2 and 5 but not for 3, 4 which contradicts to our assumption. Thus \( \varphi(H_5) \neq 5 \).

Hence we can color the graph by four colors. For b-coloring consider the color class \( c = \{1, 2, 3, 4\} \) and to assign the proper coloring we define the color function as \( f : V(H_5) \to \{1, 2, 3, 4\} \) as \( f(v) = 4, f(v_1) = 1, f(v_2) = 2, f(v_3) = 3, f(v_4) = 2, f(u_1) = 3, f(u_2) = 1, f(u_3) = 2, f(u_4) = 2, f(u_5) = 3 \). This proper coloring gives the color dominating vertices as \( cdv(1) = v_1, cdv(2) = v_2, cdv(3) = v_3, cdv(4) = v \). Thus \( \varphi(H_5) = 4 \).

**Case 4:** For \( n = 6 \)

For the graph \( H_6, |V(H_6)| = 13 \) and \( |E(H_6)| = 18 \). Also \( m(H_6) = 5 \). Then by Proposition 3.3.4 \( \varphi(H_6) \leq 5 \). Suppose that \( H_6 \) does have a b-chromatic 5-coloring. Now consider the color class \( c = \{1, 2, 3, 4, 5\} \) and to assign the proper coloring we define the color function as \( f : V(H_6) \to \{1, 2, 3, 4, 5\} \) as \( f(v) = 5, f(v_1) = 1, f(v_2) = 2, f(v_3) = 3, f(v_4) = 4, f(v_5) = 2, f(v_6) = 4, f(u_1) = 3, f(u_2) = 4, f(u_3) = 1, f(u_4) = 1, f(u_5) = 1, f(u_6) = 1 \). This proper coloring gives the color dominating vertices as \( cdv(1) = v_1, cdv(2) = v_2, cdv(3) = v_3, cdv(4) = v_4, cdv(5) = v \). Thus \( \varphi(H_6) = 5 \).

**Case 5:** For \( n \geq 7 \)

In this case, \( |V(H_7)| = 15 \) and \( |E(H_7)| = 21 \). Also \( m(H_7) = 5 \). Then by Proposition 3.3.4 \( \varphi(H_7) \leq 5 \). Suppose that \( H_7 \) does have a b-chromatic 5-coloring. Now consider the color class \( c = \{1, 2, 3, 4, 5\} \) and to assign the proper coloring we define the color function as \( f : V(H_7) \to \{1, 2, 3, 4, 5\} \) as \( f(v) = 5, f(v_1) = 4, f(v_2) = 1, f(v_3) = 2, f(v_4) = 3, f(v_5) = 4, f(v_6) = 2, f(v_7) = 1, f(u_1) = 1, f(u_2) = 3, f(u_3) = 4, f(u_4) = 1, f(u_5) = 1, f(u_6) = 1, f(u_7) = 2 \). This proper coloring gives the color dominating vertices as \( cdv(1) = v_2, cdv(2) = v_3, cdv(3) = v_4, cdv(4) = v_5, cdv(5) = v \). Thus \( \varphi(H_7) = 5 \).
For $n > 7$

We repeat the colors used for the graph $H_7$ for the vertices 
\{v, v_1, v_2, v_3, v_4, v_5, v_7, u_1, u_2, u_3, u_4, u_5, u_6, u_7\} and for the remaining vertices assign the colors as $f(v) = 5$, $f(v_{2k+6}) = 2$, $f(v_{2k+7}) = 1$, $f(u_{2k+6}) = 1$, $f(u_{2k+7}) = 2; k \in \mathbb{N}$. Hence $\phi(H_n) = 5$, $n \geq 7$.

Thus, $\varphi(H_n) = \begin{cases} 
4, & n = 3 \\
5, & n = 4 \\
4, & n = 5 \\
5, & n = 6 \\
5, & n \geq 7. 
\end{cases}$

Illustration 4.2.3. A b-coloring of $H_5$ using four colors is shown in FIGURE 4.1.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{H5_b-coloring.png}
\caption{H$_5$ and its b-coloring}
\end{figure}

Theorem 4.2.4. $H_n$ is b-continuous.

Proof. To prove this result we continue with the terminology and notations used in Lemma 4.2.1 as well as Theorem 4.2.2 and consider the following cases.

Case 1: $n = 3$

In this case $H_3$ is b-continuous as $\chi(H_3) = \varphi(H_3) = 4$. 
Case 2: $n = 4$

By Lemma 4.2.1, $\chi(H_4) = 3$ and by Theorem 4.2.2, $\varphi(H_4) = 5$. It is obvious that b-coloring for $H_4$ is possible using the number of colors $K = 3, 5$. Now for $K = 4$ the b-coloring of the graph $H_4$ is as follows. Consider the color class $c = \{1, 2, 3, 4\}$ and to assign the proper coloring we define the color function $f : V(H_4) \rightarrow \{1, 2, 3, 4\}$ as $f(v) = 4$, $f(v_1) = 1$, $f(v_2) = 2$, $f(v_3) = 3$, $f(v_4) = 2$, $f(u_1) = 3$, $f(u_2) = 1$, $f(u_3) = 1$, $f(u_4) = 1$. This proper coloring gives the color dominating vertices as $cdv(1) = v_1$, $cdv(2) = v_2$, $cdv(3) = v_3$, $cdv(4) = v$. Thus $H_4$ is b-colorable using four colors. Hence b-coloring exists for every integer $K$ satisfying $\chi(H_4) \leq K \leq \varphi(H_4)$ (Here $K = 3, 4, 5$). Consequently $H_4$ is b-continuous.

Case 3: $n = 5$

In this case the graph $H_5$ is b-continuous as $\chi(H_5) = \varphi(H_5) = 4$.

Case 4: $n = 6$

In this case by Lemma 4.2.1, $\chi(H_6) = 3$ and by Theorem 4.2.2, $\varphi(H_6) = 5$. It is obvious that b-coloring for $H_6$ is possible using the number of colors $K = 3, 5$. Now for $K = 4$ the b-coloring of the graph $H_6$ is as follows. Consider the color class $c = \{1, 2, 3, 4\}$ and to assign the proper coloring we define the color function $f : V(H_6) \rightarrow \{1, 2, 3, 4\}$ as $f(v) = 4$, $f(v_1) = 1$, $f(v_2) = 2$, $f(v_3) = 3$, $f(v_4) = 1$, $f(v_5) = 2$, $f(v_6) = 3$, $f(u_1) = 3$, $f(u_2) = 1$, $f(u_3) = 1$, $f(u_4) = 2$, $f(u_5) = 1$, $f(u_6) = 1$. This proper coloring gives the color dominating vertices as $cdv(1) = v_1$, $cdv(2) = v_2$, $cdv(3) = v_3$, $cdv(4) = v$. Thus $H_6$ is four colorable. Hence b-coloring exists for every integer $K$ satisfying $\chi(H_6) \leq K \leq \varphi(H_6)$ (Here $K = 3, 4, 5$). Hence $H_6$ is b-continuous.

Case 5: $n \geq 7$

For $n = 7$, $\chi(H_7) = 4$ by Lemma 4.2.1 and $\varphi(H_7) = 5$ by Theorem 4.2.2. It is obvious that b-coloring for $H_7$ is possible using the number of colors $K = 4, 5$. Hence b-coloring exists for every integer $K$ satisfying $\chi(H_7) \leq K \leq \varphi(H_7)$ (Here $K = 4, 5$).

For odd $n > 7$

In this case the graph $H_n$ is obviously b-continuous from $\chi(H_n) \leq K \leq \varphi(H_n)$ as $\chi(H_n) = 4$ and $\varphi(H_n) = 5$. 
For even $n > 7$

In this case we repeat the color assignment as in case $n = 6$ discussed above for the vertices $\{v, v_1, v_2, v_3, v_4, v_5, u_1, u_2, u_3, u_4, u_5, u_6\}$ and for the remaining vertices give the color as follows $f(v) = 4$, $f(v_{2k+5}) = 1$, $f(v_{2k+6}) = 2$, $f(u_{2k+5}) = 2$, $f(u_{2k+6}) = 1; k \in \mathbb{N}$. Hence $H_n$ is b-continuous.

Illustration 4.2.5. For the graph $H_6$, $\chi(H_6) = 3$ and $\varphi(H_6) = 5$. According to the definition of b-continuity $\chi(H_6) \leq K \leq \varphi(H_6) \Rightarrow 3 \leq K \leq 5$. The b-colorings using three, four and five colors are shown in Figure 4.2.

![Figure 4.2: H6 and its b-colorings](image)

Lemma 4.2.6. For the closed helm

$$\chi(CH_n) = \begin{cases} 3, & n \text{ even} \\ 4, & n \text{ odd} \end{cases}$$
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Proof. For $CH_n$, the edge set of $CH_n$ is defined as $E(CH_n) = E(H_n) \cup \{u_1u_2, u_2u_3, \ldots, u_{n-1}u_n\}$ and vertex set of $CH_n$ is $V(CH_n) = V(H_n)$ where $\nu$ be the apex, $v_1, v_2, \ldots, v_n$ be the vertices of degree four and $u_1, u_2, \ldots, u_n$ be the vertices of degree three. Also $|V(CH_n)| = 2n + 1$ and $|E(CH_n)| = 4n$. To prove this result we consider the following cases.

Case 1: $n$ even

In this case the $CH_n$ contains an odd cycles then by Proposition 3.3.3, $\chi(CH_n) \geq 3$. Now for proper coloring of $CH_n$, we need to assign color to only $\nu$, as $CH_n - \nu$ is 2-colorable. Hence $\chi(CH_n) = 3$.

Case 2: $n$ odd

In this case $CH_n$ contains $C_3$. Then by Proposition 3.3.3, $\chi(CH_n) \geq 3$. As $\nu$ is one of the vertex of $C_3$ and we have used already 3-colors for proper coloring. Therefore the fourth color is needed for $\nu$. Hence $\chi(CH_n) = 4$.

Therefore, $\chi(CH_n) = \begin{cases} 3, & n \text{ even} \\ 4, & n \text{ odd} \end{cases}$

Theorem 4.2.7.

$\varphi(CH_n) = \begin{cases} 4, & n = 3 \\ 5, & n = 4 \\ 4, & n = 5 \\ 5, & n = 6 \\ 5, & n \geq 7 \end{cases}$

Proof. To prove the result we continue with the terminology and notations used in Lemma 4.2.6 and consider the following cases.

Case 1: For $n = 3$

A closed helm $CH_3$ has, $|V(CH_3)| = 7$ and $|E(CH_3)| = 12$. Also $m(CH_3) = 4$. Then by Proposition 3.3.4, $\varphi(CH_3) \leq 4$. Further more the graph contains $K_3$, $\varphi(CH_3) \geq 3$.

We suppose that $CH_3$ has b-coloring using four colors. Now consider the set of colors $c = \{1, 2, 3, 4\}$ and to assign the proper coloring we define the color function $f : V(CH_3) \to \{1, 2, 3, 4\}$ as $f(\nu) = 4$, $f(v_1) = 1$, $f(v_2) = 2$, $f(v_3) = 3$, $f(u_1) = 2$, $f(u_2) = 1$, $f(u_3) = 4$. This proper coloring gives the color dominating vertices as
Hence we can color the graph by four colors. For b-coloring consider the color class to our assumption. Thus \( \phi(CH_3) = 4 \).

**Case 2:** For \( n = 4 \)

In this case, \(|V(CH_4)| = 9, |E(CH_4)| = 16 \) and \( m(CH_4) = 5 \). As \( CH_4 \) contains exactly 5 vertices of degree \( m(CH_4) - 1 \) then \( CH_4 \) is a tight graph. Also \( \chi(CH_4) = m(CH_4) = 5 \).

Then by Proposition 3.3.6, \( \phi(CH_4) = 5 \).

For b-coloring consider the color class \( c = \{1, 2, 3, 4, 5\} \) and to assign the proper coloring we define the color function as \( f : V(CH_4) \rightarrow \{1, 2, 3, 4, 5\} \) as \( f(v) = 5, f(v_1) = 1, f(v_2) = 2, f(v_3) = 3, f(v_4) = 4, f(u_1) = 3, f(u_2) = 4, f(u_3) = 1, f(u_4) = 2 \).

This proper coloring gives the color dominating vertices as \( cdv(1) = v_1, cdv(2) = v_2, cdv(3) = v_3, cdv(4) = v_4, cdv(5) = v \). Thus \( \phi(CH_4) = 5 \).

**Case 3:** For \( n = 5 \)

In this case \(|V(CH_5)| = 11 \) and \(|E(CH_5)| = 20 \). Also \( m(CH_5) = 5 \). Then by Proposition 3.3.4, \( \phi(CH_5) \leq 5 \). Suppose that \( CH_5 \) does have a b-chromatic 5-coloring. Now consider the color class \( c = \{1, 2, 3, 4, 5\} \) and to assign the proper coloring we define the color function as \( f : V(CH_5) \rightarrow \{1, 2, 3, 4, 5\} \) as \( f(v) = 5, f(v_1) = 4, f(v_2) = 1, f(v_3) = 2, f(v_4) = 3, f(u_1) = 3, f(u_2) = 4, f(u_3) = 1, f(u_4) = 4, f(u_5) = 2 \) which in turn forces to assign \( f(v_5) = 1 \) or 2. This proper coloring gives the color dominating vertices for color classes 1, 2 and 5 but not for color classes 3, 4 which is a contradiction to our assumption. Thus \( \phi(CH_5) \neq 5 \).

Hence we can color the graph by four colors. For b-coloring consider the color class \( c = \{1, 2, 3, 4\} \) and to assign the proper coloring we define the color function as \( f : V(CH_5) \rightarrow \{1, 2, 3, 4\} \) as \( f(v) = 4, f(v_1) = 1, f(v_2) = 2, f(v_3) = 3, f(v_4) = 1, f(v_5) = 2, f(u_1) = 3, f(u_2) = 1, f(u_3) = 2, f(u_4) = 3, f(u_5) = 1 \). This proper coloring gives the color dominating vertices as \( cdv(1) = v_1, cdv(2) = v_2, cdv(3) = v_3, cdv(4) = v \). Thus \( \phi(CH_5) = 4 \).

**Case 4:** For \( n = 6 \)

For the graph \( CH_6 \), \(|V(CH_6)| = 13 \) and \(|E(CH_6)| = 24 \). Also \( m(CH_6) = 5 \). Then by Proposition 3.3.4, \( \phi(CH_6) \leq 5 \). Suppose that \( CH_6 \) does have a b-chromatic 5-coloring.

Now consider the color class \( c = \{1, 2, 3, 4, 5\} \) and to assign the proper coloring we define the color function as \( f : V(CH_6) \rightarrow \{1, 2, 3, 4, 5\} \) as \( f(v) = 5, f(v_1) = 1, f(v_2) = 2, f(v_3) = 3, f(v_4) = 1, f(v_5) = 2, f(v_6) = 4, f(u_1) = 3, f(u_2) = 1, f(u_3) = 2, f(u_4) = 3, f(u_5) = 1, f(u_6) = 4 \). This proper coloring gives the color dominating vertices as \( cdv(1) = v_1, cdv(2) = v_2, cdv(3) = v_3, cdv(4) = v_4, cdv(5) = v_5, cdv(6) = v \). Thus \( \phi(CH_6) = 5 \).
2, f(v_3) = 3, f(v_4) = 4, f(v_5) = 1, f(v_6) = 4, f(u_1) = 3, f(u_2) = 4, f(u_3) = 1, f(u_4) = 2, f(u_5) = 3, f(u_6) = 2. This proper coloring gives the color dominating vertices as cdv(1) = v_1, cdv(2) = v_2, cdv(3) = v_3, cdv(4) = v_4, cdv(5) = v. Thus φ(CH_6) = 5.

**Case 5:** For \( n \geq 7 \)

In this case \( |V(CH_7)| = 15 \) and \( |E(CH_7)| = 28 \). Also \( m(CH_7) = 5 \). Then by Proposition 3.3.4, \( \phi(CH_7) \leq 5 \). Suppose that \( CH_7 \) does have a b-chromatic 5-coloring. Now consider the color class \( c = \{1, 2, 3, 4, 5\} \) and to assign the proper coloring we define the color function as \( f : V(CH_7) \to \{1, 2, 3, 4, 5\} \) as \( f(v) = 5, f(v_1) = 4, f(v_2) = 1, f(v_3) = 2, f(v_4) = 3, f(v_5) = 4, f(v_6) = 1, f(v_7) = 3, f(u_1) = 1, f(u_2) = 3, f(u_3) = 4, f(u_4) = 1, f(u_5) = 2, f(u_6) = 3, f(u_7) = 4 \). This proper coloring gives the color dominating vertices as cdv(1) = v_2, cdv(2) = v_3, cdv(3) = v_4, cdv(4) = v_5, cdv(5) = v. Thus \( \phi(CH_7) = 5 \).

**For \( n > 7 \)**

We repeat the colors as in the graph \( CH_7 \) for the vertices \( \{v, v_1, v_2, v_3, v_4, v_5, v_6, v_7, u_1, u_2, u_3, u_4, u_5, u_6, u_7, \} \) and for the remaining vertices assign the colors as \( f(v) = 5, f(v_{2k+6}) = 1, f(v_{2k+7}) = 2, f(u_{2k+6}) = 2, f(u_{2k+7}) = 4; k \in N \). Hence \( \phi(CH_n) = 5, n \geq 7 \).

\[
\phi(CH_n) = \begin{cases} 
4, & n = 3 \\
5, & n = 4 \\
4, & n = 5 \\
5, & n = 6 \\
5, & n \geq 7.
\end{cases}
\]

Therefore, \( \phi(CH_n) = \)
Illustration 4.2.8. A b-coloring of $CH_4$ using five colors is shown in FIGURE 4.3.

**Theorem 4.2.9.** $CH_n$ is b-continuous.

**Proof.** To prove this result we continue with the terminology and notations used in Lemma 4.2.6 as well as Theorem 4.2.7 and consider the following cases.

**Case 1:** $n = 3$

In this case the graph $CH_3$ is b-continuous as $\chi(CH_3) = \varphi(CH_3) = 4$.

**Case 2:** $n = 4$

By Lemma 4.2.6, $\chi(CH_4) = 3$ and by Theorem 4.2.7, $\varphi(CH_4) = 5$. It is obvious that b-coloring for $CH_4$ is possible using the number of colors $K = 3, 5$. Now for $K = 4$ the b-coloring of the graph $CH_4$ is as follows. Consider the color class $c = \{1, 2, 3, 4\}$ and to assign the proper coloring we define the color function $f : V(CH_4) \rightarrow \{1, 2, 3, 4\}$ as $f(v) = 4$, $f(v_1) = 1$, $f(v_2) = 2$, $f(v_3) = 3$, $f(v_4) = 2$, $f(u_1) = 3$, $f(u_2) = 4$, $f(u_3) = 1$, $f(u_4) = 4$. This proper coloring gives the color dominating vertices as $cdv(1) = v_1$, $cdv(2) = v_2$, $cdv(3) = v_3$, $cdv(4) = v$. Thus $CH_4$ is four colorable. Hence b-coloring exists for every integer $K$ satisfying $\chi(CH_4) \leq K \leq \varphi(CH_4)$ (Here $K = 3, 4, 5$). Consequently $CH_4$ is b-continuous.

**Case 3:** $n = 5$

In this case the graph $CH_5$ is b-continuous as $\chi(CH_5) = \varphi(CH_5) = 4$. 

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**FIGURE 4.3:** $CH_4$ and its b-coloring
Case 4: $n = 6$

By Lemma 4.2.6, $\chi(CH_6) = 3$ and by Theorem 4.2.7, $\varphi(CH_6) = 5$. It is obvious that b-coloring for $CH_6$ is possible using the number of colors $K = 3$ and $K = 5$. Now for $K = 4$ the b-coloring of the graph $CH_6$ is as follows. Consider the color class $c = \{1, 2, 3, 4\}$ and to assign the proper coloring we define the color function $f : V(CH_6) \rightarrow \{1, 2, 3, 4\}$ as $f(v) = 4$, $f(v_1) = 1$, $f(v_2) = 2$, $f(v_3) = 3$, $f(v_4) = 1$, $f(v_5) = 2$, $f(v_6) = 3$, $f(u_1) = 3$, $f(u_2) = 1$, $f(u_3) = 2$, $f(u_4) = 3$, $f(u_5) = 1$, $f(u_6) = 2$. This proper coloring gives the color dominating vertices as $cdv(1) = v_1$, $cdv(2) = v_2$, $cdv(3) = v_3$, $cdv(4) = v$. Thus $CH_6$ is four colorable. Hence b-coloring exists for every integer $K$ satisfying $\chi(CH_6) \leq K \leq \varphi(CH_6)$. Hence $CH_6$ is b-continuous.

Case 5: $n \geq 7$

For $n = 7$, from Lemma 4.2.6, we have $\chi(CH_7) = 4$ and by Theorem 4.2.7, $\varphi(CH_7) = 5$. It is obvious that b-coloring for $CH_7$ is possible using the number of colors $K = 4, 5$. Hence b-coloring exists for every integer $K$ satisfying $\chi(CH_7) \leq K \leq \varphi(CH_7)$ (Here $K = 4, 5$).

For even $n > 7$

In this case we repeat the color assignment as in case $n = 6$ discussed above for the vertices $\{v, v_1, v_2, v_3, v_4, v_5, v_6, u_1, u_2, u_3, u_4, u_5, u_6\}$ and for the remaining vertices give the color as follows $f(v) = 4$, $f(v_{2k+5}) = 2$, $f(v_{2k+6}) = 3$, $f(u_{2k+5}) = 1$, $f(u_{2k+6}) = 2$ where $k \in N$. Hence $CH_n$ is b-continuous. ■
Illustration 4.2.10. For the graph $CH_6$, $\chi(CH_6) = 3$ and $\varphi(CH_6) = 5$. According to the definition of b-continuity $\chi(CH_6) \leq K \leq \varphi(CH_6) \Rightarrow 3 \leq K \leq 5$. The b-colorings using three, four and five colors are shown in Figure 4.4.

In accordance with helm and closed helm considered in above Theorem 4.2.2 and Theorem 4.2.7 we state the following corollary.

Corollary 4.2.11.

$$S_b(H_n) = S_b(CH_n) = \begin{cases} \{4\}, & n = 3, 5 \\ \{3, 4, 5\}, & n = 4, 6 \\ \{4, 5\}, & odd \ n \geq 7 \\ \{3, 4, 5\}, & even \ n > 7. \end{cases}$$

Proof. The proof is straightforward. □

Lemma 4.2.12.

$$\chi(Fl_n) = \begin{cases} 4, & n \ odd \\ 3, & n \ even. \end{cases}$$

Proof. Let $v$ be the apex, $v_1, v_2, \ldots, v_n$ be the vertices of degree 4 and $u_1, u_2, \ldots, u_n$ be the vertices of degree 2 in $Fl_n$. Then $|V(Fl_n)| = 2n + 1$ and $|E(Fl_n)| = 4n$. We consider following two cases.
Case 1: $n$ even
In this case $Fl_n$ contains even $W_n$ as an induced subgraph. Since $\chi(G) = 3 \Rightarrow \chi(Fl_n) = 3$.

Case 2: $n$ odd
In this case $Fl_n$ contains odd $W_n$ as an induced subgraph. Since $\chi(G) = 4 \Rightarrow \chi(Fl_n) = 4$.

Thus, $\chi(Fl_n) = \begin{cases} 
4, & n \text{ odd} \\
3, & n \text{ even}.
\end{cases}$ ■

We continue with the terminology and notations used in Lemma 4.2.12 to prove the next two results.

**Theorem 4.2.13.**

$$
\varphi(Fl_n) = \begin{cases} 
4, & n = 3 \\
5, & n = 4 \\
4, & n = 5 \\
5, & n \geq 6
\end{cases}
$$

**Proof.** To prove the result we consider following four cases.

Case 1: $n = 3$
By Lemma 4.2.12, $\chi(Fl_3) = 4$ and $m(Fl_3) = 4$. Then by Proposition 3.3.4, $4 \leq \varphi(Fl_3) \leq 5$. If $Fl_3$ does have b-chromatic 5-coloring then $f(v) = 5$, $f(v_1) = 4$, $f(v_2) = 2$, $f(v_3) = 3$, $f(v_4) = 4$, $f(u_1) = 3$, $f(u_2) = 4$, $f(u_3) = 1$, $f(u_4) = 2$.

Case 2: $n = 4$
$Fl_4$ has at least five vertices of degree four. Thus $\varphi(Fl_4) \leq m(Fl_4) = 5$. If possible suppose $\varphi(Fl_4) = 5$. By assigning proper 5-coloring to the vertices as $f(v) = 5$, $f(v_1) = 1$, $f(v_2) = 2$, $f(v_3) = 3$, $f(v_4) = 4$, $f(u_1) = 3$, $f(u_2) = 4$, $f(u_3) = 1$, $f(u_4) = 2$.

This proper coloring gives the color dominating vertices as $cdv(1) = v_1$, $cdv(2) = v_2$, $cdv(3) = v_3$, $cdv(4) = v_4$, $cdv(5) = v$. Thus, $\varphi(Fl_4) = 5$.

Case 3: $n = 5$
By Lemma 4.2.12, $\chi(Fl_5) = 4$ and also $m(Fl_5) = 4$. Then by Proposition 3.3.4, $4 \leq \varphi(Fl_5) \leq 5$. If $Fl_5$ does have b-chromatic 5-coloring then $f(v) = 5$, $f(v_1) = 4$, $f(v_2) = 2$, $f(v_3) = 3$, $f(v_4) = 4$, $f(u_1) = 3$, $f(u_2) = 4$, $f(u_3) = 1$, $f(u_4) = 2$.
3, \( f(v_2) = 1, f(v_3) = 2, f(v_4) = 3, f(v_5) = 4, f(u_1) = 2, f(u_2) = 4, f(u_3) = 4, f(u_4) = 1, f(u_5) = 1 \). This proper coloring gives the color dominating vertices for the color classes 1, 2, 3, 5 but not for color class 4. Similarly all other proper colorings using 5 colors will generate color dominating vertices for the color classes 1, 2, 3, 5 but not for color class 4. Hence \( \varphi(Fl_5) \neq 5 \). Thus \( \varphi(Fl_5) \leq 4 \). As \( Fl_5 \) contains \( W_5 \) as a subgraph \( \chi(W_5) = 4 \), we have \( \varphi(Fl_5) \geq \chi(Fl_5) \Rightarrow \varphi(Fl_5) \geq 4 \). Hence, \( \varphi(Fl_5) = 4 \).

Case 4: \( n \geq 6 \)

Now by Lemma 4.2.12, \( \chi(Fl_n) = 3 \) and \( m(Fl_n) = 5 \). Then by Proposition 3.3.4, we have \( 3 \leq \varphi(Fl_n) \leq 5 \). Since \( Fl_n \) has at least five vertices of degree four. Thus \( \varphi(Fl_n) \leq m(Fl_n) = 5 \). By assigning proper coloring to the vertices as \( f(v) = 5, f(v_1) = 4, f(v_2) = 1, f(v_3) = 2, f(v_4) = 3, f(v_5) = 4, f(v_n) = 2, f(u_1) = 1, f(u_2) = 3, f(u_3) = 4, f(u_4) = 1, f(u_5) = 1, f(u_n) = 1 \). For remaining vertices we proceed with any proper coloring. This proper coloring gives the color dominating vertices \( v_2, v_3, v_4, v_5, v \) for the color classes 1, 2, 3, 4, 5 respectively. Thus, \( \varphi(Fl_n) = 5 \).

Hence, \( \varphi(Fl_n) = \begin{cases} 
4, & n = 3 \\
5, & n = 4 \\
4, & n = 5 \\
5, & n \geq 6.
\end{cases} \) ■
**Illustration 4.2.14.** A b-coloring of $Fl_4$ using five colors is shown in **FIGURE 4.5.**

![Figure 4.5: Fl and its b-coloring](image)

**Theorem 4.2.15.** $Fl_n$ is b-continuous.

**Proof.** To prove the result we consider the following cases.

**Case 1: $n = 3$**

In this case the graph $Fl_3$ is b-continuous as $\chi(Fl_3) = \varphi(Fl_3) = 4$.

**Case 2: $n = 4$**

In this case by Lemma 4.2.12, $\chi(Fl_4) = 3$ and by Theorem 4.2.13, $\varphi(Fl_4) = 5$. It is obvious that b-coloring for the graph $Fl_4$ is possible using the number of colors $K = 3, 5$.

Now for $K = 4$ the b-coloring for the graph $Fl_4$ is as follows. Consider the color class $c = \{1, 2, 3, 4\}$. To assign the proper coloring to the vertices define the color function $f : V(Fl_4) \to \{1, 2, 3, 4\}$ as $f(v) = 4, f(v_1) = 1, f(v_2) = 2, f(v_3) = 3, f(v_4) = 2, f(u_1) = 3, f(u_1) = 3, f(u_2) = 1, f(u_3) = 1, f(u_4) = 3$. This proper coloring gives the color dominating vertices $v, v_1, v_2, v_3$ for the color classes $1, 2, 3, 4$ respectively. Thus $Fl_4$ is four colorable. Hence b-coloring exists for every integer $K$ satisfying $\chi(Fl_4) \leq K \leq \varphi(Fl_4)$ (Here $K = 3, 4, 5$). Thus, $Fl_4$ is b-continuous.

**Case 3: $n = 5$**

For $Fl_5$ by Lemma 4.2.12, $\chi(Fl_5) = 4$ and by Theorem 4.2.13, $\varphi(Fl_5) = 4$. Hence b-coloring exists for every integer $K$ satisfying $\chi(Fl_5) \leq K \leq \varphi(Fl_5)$ (Here $K = 4$). Thus, $Fl_5$ is b-continuous.
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Case 4: $n \geq 6$

For $n = 6$, By Lemma 4.2.12, $\chi(Fl_6) = 3$ and by Theorem 4.2.13, $\varphi(Fl_6) = 5$. It is obvious that b-coloring for the graph $Fl_6$ is possible using the number of colors $K = 3, 5$.

Now for $K = 4$ repeat the colors as in the case $n = 4$ and for the remaining vertices we proceed with any proper coloring. This proper coloring gives the color dominating vertices $v, v_1, v_2, v_3$ for the color classes 1, 2, 3, 4 respectively. Thus, $Fl_6$ is $b$-continuous.

When $n > 6$

Repeat the colors as in $Fl_6$ for the vertices $v_1, v_2, \ldots, v_6, u_1, u_2, \ldots, u_6, v$ and assign the proper coloring to the remaining vertices. Thus, $Fl_n$ is $b$-continuous for all $n > 6$. ■

4.3 Duplication of vertices in wheel $W_n$ and $b$-coloring

Theorem 4.3.1. Let $G_1$ be the graph obtained from graph $G$ by duplication of vertices(vertex) then $\chi(G) = \chi(G_1)$.

Proof. Let $v \in V(G)$ be an arbitrary vertex of $G$ and $v' \in V(G_1)$ be its duplicated vertex. As $N(v) = N(v')$ in $G_1$ and $v$ and $v'$ are independent vertices we can assign the same colors to $v$ and $v'$. Thus no extra color is required for proper coloring of $G_1$. As all the duplicated vertices are independent in $G_1$, this argument can be extended in the case when arbitrary number of vertices are duplicated . Hence $\chi(G) = \chi(G_1)$. ■

Theorem 4.3.2. Let $G$ be the graph obtained by duplicating all the rim vertices in $W_n$ then

$$\varphi(G) = \begin{cases} 
4, & n = 3 \\
3, & n = 4 \\
5, & n = 5, 6, 8 \\
6, & n = 7 \\
6, & n \geq 9.
\end{cases}$$
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Proof. Let $v_1, v_2, \ldots, v_n$ be the rim vertices and $v$ be the apex vertex of $W_n$ and $G$ be the graph obtained by duplication of all the rim vertices of $W_n$. Let $v'_1, v'_2, \ldots, v'_n$ be the duplicated vertices corresponding to $v_1, v_2, \ldots, v_n$. Then $|V(G)| = 2n + 1$ and $|E(G)| = 5n$.

To define proper coloring we consider the following cases.

Case 1: $n = 3$

In this case we have $V(G) = \{v_1, v_2, v_3, v'_1, v'_2, v'_3, u\}$ and $|V(G)| = 7$. More precisely $G$ has three vertices of degree three, three vertices of degree five and a vertex of degree six. Then by Proposition 3.3.1, $\varphi(G) \leq 7$ as $\Delta(G) = 6$.

If $\varphi(G) = 7$, then according to Proposition 3.3.2, $G$ must have seven vertices of degree six which is not possible as there is only one vertex of degree six. Hence $\varphi(G) \neq 7$.

If $\varphi(G) = 6$ then according to Proposition 3.3.2, the graph $G$ must have six vertices of degree five which is not possible as there are only three vertices of degree five and the remaining vertices are of degree three. Hence $\varphi(G) \neq 6$.

We claim that $\varphi(G) \neq 5$ because to achieve $\varphi(G) = 5$ we need minimum five vertices of degree four, which is not possible by Proposition 3.3.2, as there are only three vertices of degree five and the remaining one vertex is of degree six while three vertices are of degree three. Hence $\varphi(G) \neq 5$.

If $\varphi(G) = 4$ then according to Proposition 3.3.2, the graph $G$ must have four vertices of degree three, which is possible for $G$. For $b$-coloring consider the color class $c = \{1, 2, 3, 4\}$ and to assign the proper coloring to the vertices define the color function $f : V(G) \to \{1, 2, 3, 4\}$ as $f(v_1) = f(v'_1) = 1$, $f(v_2) = f(v'_2) = 2$, $f(v_3) = f(v'_3) = 3$, $f(v) = 4$. This proper coloring gives $cdv(1) = v'_1$, $cdv(2) = v'_2$, $cdv(3) = v'_3$, $cdv(4) = v$. Hence $\varphi(G) = 4$.

Case 2: $n = 4$

For graph $G$ we have $V(G) = \{v_1, v_2, v_3, v_4, v'_1, v'_2, v'_3, v'_4, u\}$ and $|V(G)| = 9$. More precisely graph $G$ has four vertices of degree three, four vertices of degree five and one vertex of degree eight. Then by Proposition 3.3.2, we have $\varphi(G) \leq 9$ as $\Delta(G) = 8$. If $\varphi(G) = 9, 8, 7$ then the respective graphs do not have the required number of $m$-degree vertices. Therefore it is not possible to obtain $b$-coloring with said number of colors.

If $\varphi(G) = 6$ then according to Proposition 3.3.2, $G$ must have six vertices of degree
five, which is not possible as there are only four vertices of degree five, four vertices of degree three and one vertex of degree eight. Hence \( \phi(G) \neq 6 \)

If \( \phi(G) = 5 \), then according to Proposition 3.3.2, the graph \( G \) must have five vertices of degree four that is not possible as there is no vertex of degree four. Hence \( \phi(G) \neq 5 \)

If \( \phi(G) = 4 \) then by Proposition 3.3.2, \( G \) must have four vertices of degree three which is possible. But due to nature of the graph \( G \) any proper coloring with four colors have at least one color class which does not have color dominating vertices hence it will not be \( b \)-coloring for the graph \( G \). Hence \( \phi(G) \neq 4 \). Thus we can color the graph by three colors.

For \( b \)-coloring consider the color class \( c = \{1, 2, 3\} \) and to assign the proper coloring to the vertices define the color function \( f : V(G) \rightarrow \{1, 2, 3\} \) as \( f(v_1) = f(v'_1) = 1 \), \( f(v_2) = f(v'_2) = 2 \), \( f(v_3) = f(v'_3) = 1 \), \( f(v_4) = f(v'_4) = 2 \), \( f(v) = 3 \). This proper coloring gives \( cdv(1) = v'_1, \ cdv(2) = v'_2, \ cdv(3) = v \). Hence \( \phi(G) = 3 \).

**Case 3:** When \( n = 5, 6, 8 \)

**Subcase 1:** For \( n = 5 \)

In this case we have \( V(G) = \{v_1, v_2, v_3, v_4, v_5, v'_1, v'_2, v'_3, v'_4, v'_5, v\} \) and \( |V(G)| = 11 \). More precisely \( G \) has five vertices of degree three, five vertices of degree five and one vertex of degree ten. Then by Proposition 3.3.1, \( \phi(G) \leq 11 \) as \( \Delta(G) = 10 \).

If \( \phi(G) = 11, 10, 9, 8, 7 \) then the respective graphs do not have the required number of \( m \)-degree vertices. Therefore it is not possible to obtain \( b \)-coloring with said number of colors.

If \( \phi(G) = 6 \), then \( G \) must have six vertices of degree at least five which is not possible as there are only five vertices of degree five and the remaining vertices are of degree three while one vertex is of degree ten. Hence \( \phi(G) \neq 6 \)

If \( \phi(G) = 5 \), then \( G \) must have five vertices of degree at least four which is possible for the graph \( G \). Thus we can color the graph by five colors.

Now for \( b \)-coloring consider the set of colors \( c = \{1, 2, 3, 4, 5\} \) and to assign the proper coloring to the vertices define the color function \( f : V(G) \rightarrow \{1, 2, 3, 4, 5\} \) as \( f(v_1) = 4, f(v_2) = 1, f(v_3) = 2, f(v_4) = 3, f(v_5) = 2, f(v'_1) = 3, f(v'_2) = 3, f(v'_3) = 4, f(v'_4) = 4, f(v'_5) = 1, f(v) = 5 \). This proper coloring gives \( cdv(1) = v_2, cdv(2) = v_3, cdv(3) = \)}
\(v_4, \text{cdv}(4) = v_1, \text{cdv}(5) = v\). Hence \(\varphi(G) = 5\).

**Subcase 2:** For \(n = 6\)

For graph \(G\) we have \(V(G) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_1', v_2', v_3', v_4', v_5', v_6', v\}\) and \(|V(G)| = 13\). More precisely \(G\) has six vertices of degree three, six vertices of degree five and the remaining vertex is of degree twelve. Then by Proposition 3.3.1, \(\varphi(G) \leq 13\) as \(\Delta(G) = 12\).

If \(\varphi(G) = 12, 11, 10, 9, 8, 7\) then the respective graphs do not have the required number of \(m\)-degree vertices so it is not possible to obtain b-coloring with said number of colors.

If \(\varphi(G) = 6\), then \(G\) must have six vertices of degree at least five which is possible. But due to the nature of the graph \(G\) any proper coloring with six colors have at least one color class which does not have color dominating vertices hence it will not be b-coloring for the graph \(G\). Thus \(\varphi(G) \neq 6\).

For b-coloring with five colors consider the color class \(c = \{1, 2, 3, 4, 5\}\) and to assign the proper coloring to vertices define the color function \(f : V(G) \to \{1, 2, 3, 4, 5\}\) as

\[f(v_1) = f(v_1') = 3, f(v_2) = 1, f(v_3) = 2, f(v_4) = 3, f(v_5) = 4, f(v_6) = 2, f(v_2') = 4, f(v_3') = 1, f(v_4') = 1, f(v_5') = 1, f(v) = 5.\]

This proper coloring gives \(\text{cdv}(1) = v_2, \text{cdv}(2) = v_3, \text{cdv}(3) = v_4, \text{cdv}(4) = v_5, \text{cdv}(5) = v\). Hence \(\varphi(G) = 5\).

**Subcase 3:** For \(n = 8\)

In this case we have \(V(G) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_1', v_2', v_3', v_4', v_5', v_6', v_7', v_8', v\}\) and \(|V(G)| = 17\). More precisely \(G\) has eight vertices of degree three, eight vertices of degree five and one vertex of degree sixteen. Then by Proposition 3.3.1, \(\varphi(G) \leq 17\) as \(\Delta(G) = 16\).

If \(\varphi(G) = 17, 16, 15, 14, 13, 12, 11, 10, 9, 8, 7\) then the respective graphs do not have the required number of \(m\)-degree vertices. Therefore it is not possible to obtain b-coloring with said number of colors.

If \(\varphi(G) = 6\), then graph \(G\) must have six vertices of degree at least five which is possible. But due to nature of the graph \(G\) any proper coloring with six colors have at least one color class which does not have color dominating vertices hence it will not be b-coloring for the graph \(G\). Thus \(\varphi(G) \neq 6\).

For b-coloring with five colors consider the color class \(c = \{1, 2, 3, 4, 5\}\) and to assign
the proper coloring to the vertices define the color function \( f : V(G) \to \{1,2,3,4,5\} \) as \( f(v_1) = f(v'_1) = 3, f(v_2) = 1, f(v_3) = 2, f(v_4) = 3, f(v_5) = 4, f(v_6) = 2, f(v_7) = 3, f(v'_2) = 4, f(v'_3) = 4, f(v'_4) = 1, f(v'_5) = 1, f(v'_6) = 1, f(v'_7) = 3, f(v_8) = f(v'_8) = 1, f(v) = 5 \). This proper coloring gives \( cdv(1) = v_2, cdv(2) = v_3, cdv(3) = v_4, cdv(4) = v_5, cdv(5) = v \). Hence \( \varphi(G) = 5 \).

**Case 4: \( n = 7 \)**

For graph \( G \) we have \( V(G) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v'_1, v'_2, v'_3, v'_4, v'_5, v'_6, v'_7, v, v'\} \) and \( |V(G)| = 15 \). More precisely \( G \) has seven vertices of degree three, seven vertices of degree five and one vertex is of degree fourteen. Then by Proposition 3.3.1, \( \varphi(G) \leq 15 \) as \( \Delta(G) = 14 \).

If \( \varphi(G) = 15, 14, 13, 12, 11, 10, 9, 8, 7 \) then the respective graphs do not have the required number of \( m \)-degree vertices. Therefore it is not possible to obtain b-coloring with said number of colors.

If \( \varphi(G) = 6 \), then according to Proposition 3.3.2, we need minimum six vertices of degree at least five which is possible. For b-coloring consider the color class \( c = \{1,2,3,4,5,6\} \) and to assign the proper coloring to the vertices define the color function \( f : V(G) \to \{1,2,3,4,5,6\} \) as \( f(v_1) = 5, f(v_2) = 1, f(v_3) = 2, f(v_4) = 3, f(v_5) = 1, f(v_6) = 4, f(v_7) = 2, f(v'_1) = 3, f(v'_2) = 4, f(v'_3) = 4, f(v'_4) = 5, f(v'_5) = 5, f(v'_6) = 4, f(v'_7) = 3, f(v) = 6 \). This proper coloring gives \( cdv(1) = v_2, cdv(2) = v_3, cdv(3) = v_4, cdv(4) = v_6, cdv(5) = v_1, cdv(6) = v \) which confirms that \( \varphi(G) = 6 \).

**Case 5: \( n \geq 9 \)**

For \( n = 9 \), \( V(G) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_1', v_2', v_3', v_4', v_5', v_6', v_7', v_8', v_9', v\} \) and \( |V(G)| = 19 \). More precisely \( G \) has nine vertices of degree five, nine vertices of degree three and one vertex of degree eighteen. Then by Proposition 3.3.1, \( \varphi(G) \leq 19 \) as \( \Delta(G) = 18 \).

If \( \varphi(G) = 19, 18, 17, 16, 15, 14, 13, 12, 11, 10, 9, 8, 7 \) then the respective graphs do not have the required number of \( m \)-degree vertices so it is not possible to obtain b-coloring with said number of colors.

According to Proposition 3.3.2, if \( \varphi(G) = 6 \) then we need minimum six vertices of degree at least five which is possible. For b-coloring consider the color class \( c = \{
\{1, 2, 3, 4, 5, 6\} and to assign the proper coloring to the vertices define the color function \( f : V(G) \to \{1, 2, 3, 4, 5, 6\} \) as
\[

g f(v_1) = 4, f(v_2) = 2, f(v_3) = 5, f(v_4) = 1, f(v_5) = 2, f(v_6) = 3, f(v_7) = 1, f(v_8) = 4, f(v_9) = 2, f(v_1') = 4, f(v_2') = 3, f(v_3') = 3, f(v_4') = 4, f(v_5') = 4, f(v_6') = 5, f(v_7') = 5, f(v_8') = 4, f(v_9') = 3, f(v) = 6.
\]
This proper coloring gives \( cdv(1) = v_2, cdv(2) = v_3, cdv(3) = v_4, cdv(4) = v_6, cdv(5) = v_1, cdv(6) = v \). Hence \( \varphi(G) = 6 \).

When \( n > 9 \) we repeat the colors as in the above graph \( G \) for the vertices \( \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_1', v_2', v_3', v_4', v_5', v_6', v_7', v_8', v_9'\} \) and for the remaining vertices assign the colors as follows \( f(v_i) = 1, f(v_i') = 2; \text{ when } i \text{ is even } \) and \( f(v_i) = 2, f(v_i') = 1; \text{ when } i \text{ is odd } \). Hence \( \varphi(G) = 6, n \geq 9 \).

\[
\varphi(G) = \begin{cases} 
4, & n = 3 \\
3, & n = 4 \\
5, & n = 5, 6, 8 \\
6, & n = 7 \\
6, & n \geq 9.
\end{cases}
\]

**Illustration 4.3.3.** A b-coloring of duplication of rim vertices of wheel using three colors is shown in Figure 4.6.

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**Figure 4.6:** Duplication of rim vertices of wheel \( W_4 \) and its b-coloring
**Theorem 4.3.4.** $G$ is b-continuous.

**Proof.** To prove this result we continue with the terminology and notations used in Theorem 4.3.2 and consider the following cases.

**Case 1: $n = 3$**
In this case the graph $G$ is b-continuous as $\chi(G) = \phi(G) = 4$.

**Case 2: $n = 4$**
In this case the graph $G$ is b-continuous as $\chi(G) = \phi(G) = 3$.

**Case 3: $n = 5$**
In this case by Theorem 4.3.1, and Proposition 3.3.7, we have $\chi(G) = \chi(W_5) = 4$. Also by Theorem 4.3.2, $\phi(G) = 5$. Thus b-coloring exists for every integer $K$ satisfying $\chi(G) \leq K \leq \phi(G)$ (Here $K = 4, 5$). Hence $G$ is b-continuous.

**Case 4: $n = 6$**
In this case by Theorem 4.3.1, and Proposition 3.3.7, we have $\chi(G) = \chi(W_6) = 3$. Also by Theorem 4.3.2, $\phi(G) = 5$. It is obvious that b-coloring for $G$ is possible using the number of colors $K = 3, 5$.

Now for $K = 4$ the b-coloring for the graph $G$ is as follows. Consider the color class $c = 1, 2, 3, 4$ and to assign the proper coloring to the vertices define the color function $f : V(G) \rightarrow \{1, 2, 3, 4\}$ as $f(v_1) = 1 = f(v'_1), f(v_2) = 2 = f(v'_2), f(v_3) = 3 = f(v'_3), f(v_4) = 1 = f(v'_4), f(v_5) = 2 = f(v'_5), f(v_6) = 3 = f(v'_6), f(v) = 4$.

This proper coloring gives the color dominating vertices as $cdv(1) = v_1, cdv(2) = v_2, cdv(3) = v_3, cdv(4) = v$. Thus $G$ is four colorable. Hence b-coloring exists for every integer $K$ satisfying $\chi(G) \leq K \leq \phi(G)$ (Here $K = 3, 4, 5$). Hence $G$ is b-continuous.

**Case 5: $n = 7$**
In this case by Theorem 4.3.1, and Proposition 3.3.7, we have $\chi(G) = \chi(W_7) = 4$. Also by Theorem 4.3.2, $\phi(G) = 6$. It is obvious that b-coloring for $G$ is possible using the number of colors $K = 4, 6$. Now for $K = 5$ the b-coloring for the graph $G$ is as follows. Consider the color class $c = \{1, 2, 3, 4, 5\}$ and to assign the proper coloring to the vertices define the color function $f : V(G) \rightarrow \{1, 2, 3, 4, 5\}$ as $f(v_1) = 1, f(v_2) = 2, f(v_3) = 3, f(v_4) = 1, f(v_5) = 4, f(v_6) = 2, f(v_7) = 4, f(v'_1) = 1, f(v'_2) = 4, f(v'_3) =$
Case 6: \( n = 8 \).

In this case by Theorem 4.3.1, and Proposition 3.3.7, we have \( \chi(G) = \chi(W_8) = 3 \). Also by Theorem 4.3.2, \( \varphi(G) = 5 \). It is obvious that b-coloring for \( G \) is possible using the number of colors \( K = 3, 5 \). Now for \( K = 4 \) the b-coloring for the graph \( G \) is as follows.

Consider the color class \( c = \{1, 2, 3, 4\} \) and to assign the proper coloring to the vertices define the color function \( f: V(G) \rightarrow \{1, 2, 3, 4\} \) as \( f(v_1) = 1 = f(v'_1), f(v_2) = 2 = f(v'_2), f(v_3) = 3 = f(v'_3), f(v_4) = 1 = f(v'_4), f(v_5) = 2 = f(v'_5), f(v_6) = 3 = f(v'_6), f(v_7) = 1 = f(v'_7), f(v_8) = 3 = f(v'_8), f(v) = 4 \). This proper coloring gives the color dominating vertices as \( cdv(1) = v_1, cdv(2) = v_2, cdv(3) = v_3, cdv(4) = v_5, cdv(5) = v_9 \). So the graph \( G \) is five colorable. Hence b-coloring exists for every integer \( K \) satisfying \( \chi(G) \leq K \leq \varphi(G) \) (Here \( K = 4, 5, 6 \)). Thus \( G \) is b-continuous.

Case 7: \( n = 9 \).

In this case by Theorem 4.3.1, and Proposition 3.3.7, we have \( \chi(G) = \chi(W_9) = 4 \). Also by Theorem 4.3.2, \( \varphi(G) = 6 \). It is obvious that b-coloring for \( G \) is possible using the number of colors \( K = 4, 6 \). Now for \( K = 5 \) the b-coloring for the graph \( G \) is as follows.

Consider the color class \( c = \{1, 2, 3, 4, 5\} \) and to assign the proper coloring to the vertices define the color function \( f: V(G) \rightarrow \{1, 2, 3, 4, 5\} \) as \( f(v_1) = 2, f(v_2) = 4, f(v_3) = 1, f(v_4) = 2, f(v_5) = 3, f(v_6) = 1, f(v_7) = 2, f(v_8) = 1, f(v_9) = 3, f(v'_1) = 2, f(v'_2) = 3, f(v'_3) = 3, f(v'_4) = 4, f(v'_5) = 4, f(v'_6) = 1, f(v'_7) = 2, f(v'_8) = 1, f(v'_9) = 3, f(v) = 5 \). This proper coloring gives the color dominating vertices as \( cdv(1) = v_1, cdv(2) = v_2, cdv(3) = v_3, cdv(4) = v_9, cdv(5) = v_9 \). Thus \( G \) is five colorable. Hence b-coloring exists for every integer \( K \) satisfying \( \chi(G) \leq K \leq \varphi(G) \) (Here \( K = 4, 5, 6 \)).

Case 8: \( n > 9 \)

When \( n > 9 \) we repeat the color assignment as in the case \( n = 9 \) discussed above for
the vertices \( \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v'_1, v'_2, v'_3, v'_4, v'_5, v'_6, v'_7, v'_8, v'_9, v\} \) and for the remaining vertices give the colors as follows.

\[
f(v_i) = f(v'_i) = \begin{cases} 
1, & i \text{ even} \\
3, & i \text{ odd}
\end{cases}
\]

Hence \( G \) is b-continuous.

\[\square\]

**Illustration 4.3.5.** If the graph \( G \), is obtained from duplication of rim vertices in \( W_6 \) then \( \chi(G) = 3 \) and \( \phi(G) = 5 \). According to the definition of b-continuity \( \chi(G) \leq K \leq \phi(G) \) \( \Rightarrow 3 \leq K \leq 5 \). The b-colorings using three, four and five colors are shown in Figure 4.7.
Chapter 4. The b-chromatic number of some wheel related graphs

In accordance with the graph $G$ considered in above Theorem 4.3.1 we state the following corollary.

**Corollary 4.3.6.**

$$S_b(G) = \begin{cases} 
\{4\} & n = 3 \\
\{3\} & n = 4 \\
\{4, 5\} & n = 5 \\
\{3, 4, 5\} & n = 6, 8 \\
\{4, 5, 6\} & n = 7 \\
\{4, 5, 6\} & n \geq 9.
\end{cases}$$

**Proof.** The proof is straightforward. ■

**Theorem 4.3.7.** Let $G_1$ be the graph obtained by duplicating the apex vertex in $W_n$ then

$$\varphi(G_1) = \begin{cases} 
4, & n = 3 \\
3, & n = 4 \\
4, & n \geq 5.
\end{cases}$$

**Proof.** For $W_n$, $v_1, v_2, \ldots, v_n$ be the vertices and $v$ be the apex vertex of $W_n$. Let $G_1$ be the graph obtained by duplication of the vertex $v$ of $W_n$. Let $v'$ be the duplicated vertices corresponding to $v$. Then $|V(G_1)| = n + 2$ and $|E(G_1)| = 3n$. To define the proper coloring we consider the following four cases.

**Case 1: $n = 3$**

In this case $V(G_1) = \{v_1, v_2, v_3, v, v'\}$ and $|V(G_1)| = 5$. More precisely $G_1$ has two vertices of degree three, three vertices of degree four. Then by Proposition 3.3.1, $\varphi(G_1) \leq 5$ as $\Delta(G_1) = 4$. If $\varphi(G_1) = 4$ then according to Proposition 3.3.2, the graph $G_1$ must have four vertices of degree at least three which is possible.

For b-coloring consider the color class $c = \{1, 2, 3, 4\}$ and to assign the proper coloring to the vertices define the color function $f : V(G_1) \rightarrow \{1, 2, 3, 4\}$ as $f(v_1) = 1, f(v_2) = 2, f(v_3) = 3, f(v) = 4 = f(v')$. This proper coloring gives $cdv(1) = v_1, cdv(2) = v_2, cdv(3) = v_3, cdv(4) = v$. Hence $\varphi(G_1) = 4$. 
Case 2: $n = 4$
In this case $V(G_1) = \{v_1, v_2, v_3, v_4, v, v'\}$ and $|V(G_1)| = 6$. More precisely $G_1$ has six vertices of degree four. Then by Proposition 3.3.1, $\varphi(G_1) \leq 5$ as $\Delta(G_1) = 4$. If $\varphi(G_1) = 5$ then according to Proposition 3.3.2 the graph $G_1$ must have five vertices of degree at least four which is possible. But due to the nature of graph $G_1$ any proper coloring with five colors have at least one color class which does not have color dominating vertices. Hence $G_1$ is not b-colorable with five colors. Hence $\varphi(G_1) \neq 5$.
If possible let $\varphi(G_1) = 4$ and $f(v_1) = 1, f(v_2) = 2, f(v_3) = 3, f(v) = 4$, which in turn forces us to assign $f(v_4) = 2, f(v') = 4$. This proper coloring gives the color dominating vertices for color classes 2 and 4 but not for 1 and 3 which contradicts our assumption. Thus $\varphi(G_1) \neq 4$. Hence we can color the graph by three colors.
For b-coloring consider the color class $c = \{1, 2, 3\}$ and to assign the proper coloring to the vertices define the color function $f : V(G_1) \rightarrow \{1, 2, 3\}$ as $f(v_1) = 1, f(v_2) = 2, f(v_3) = 1, f(v_4) = 2, f(v) = 3, f(v') = 3$. This proper coloring gives $cdv(1) = v_1, cdv(2) = v_2, cdv(3) = v$. Hence $\varphi(G_1) = 3$.

Case 3: $n = 5$
For graph $G_1$ we have $V(G_1) = \{v_1, v_2, v_3, v_4, v_5, v, v'\}$ and $|V(G_1)| = 7$. More precisely $G_1$ has five vertices of degree four and two vertices of degree two. Then by Proposition 3.3.1, $\varphi(G_1) \leq 6$ as $\Delta(G_1) = 5$. According to Proposition 3.3.2, if $\varphi(G_1) = 6$ then we need six vertices of degree at least five, which is not possible as there are only two vertices of degree five and the remaining vertices are of degree four. Hence $\varphi(G_1) \neq 6$.
If $\varphi(G_1) = 5$ then according to Proposition 3.3.2, the graph $G_1$ must have five vertices of degree at least four which is possible. But due to the nature of graph $G_1$ any proper coloring with five colors have at least one color class which does not have any color dominating vertex. Hence $G_1$ is not b-colorable with five colors. Hence $\varphi(G_1) \neq 5$.
Thus we can color the graph by four colors.
For b-coloring consider the color class $c = \{1, 2, 3, 4\}$ and to assign the proper coloring to the vertices define the color function $f : V(G_1) \rightarrow \{1, 2, 3, 4\}$ as $f(v_1) = 3, f(v_2) = 1, f(v_3) = 2, f(v_4) = 3, f(v_5) = 1, f(v) = 4, f(v') = 4$. This proper coloring gives $cdv(1) = v_2, cdv(2) = v_3, cdv(3) = v_4, cdv(4) = v$. Hence $\varphi(G_1) = 4$. 
**Case 4: n > 5**

When $n > 5$ we repeat the color assignment as in the case when $n = 5$ for the vertices $\{v_1, v_2, v_3, v_4, v_5, v'\}$ and for the remaining vertices assign the colors as follows.

$$f(v_i) = \begin{cases} 
2, & i \text{ is even} \\
1, & i \text{ is odd} 
\end{cases}$$

Hence $\varphi(G_1) = 4$; $n \geq 5$.

Thus $\varphi(G_1) = \begin{cases} 
4, & n = 3 \\
3, & n = 4 \\
4, & n \geq 5 
\end{cases}$

**Illustration 4.3.8.** A b-coloring of duplication of an apex vertex using four colors is shown in Figure 4.8.

![Figure 4.8: Duplication of an apex vertex of wheel $W_5$ and its b-coloring](image)

In the previous section the duplication of rim vertices and apex vertex is performed separately while the splitting graph is obtained by duplicating vertices of a graph all together. Now we prove some results for splitting graph of $W_n$.

**Lemma 4.3.9.**

$$\chi[S'(W_n)] = \begin{cases} 
3, & n \text{ even} \\
4, & n \text{ odd} 
\end{cases}$$
Proof. Let $v_1, v_2, \ldots, v_n$ be the rim vertices of wheel $W_n$ which are duplicated by the vertices $v'_1, v'_2, \ldots, v'_n$ respectively and let $v$ denotes the apex vertex of $W_n$ which is duplicated by the vertex $v'$. Let $e_1, e_2, \ldots, e_n$ be the rim edges of $W_n$. Then the resultant graph $S'[W_n]$ will have order $2(n + 1)$ and size $6n$.

Here $N(v_i) = N(v'_i)$, for $i \in N, N(v) = N(v')$ and $S'(W_n)$ contains $W_n$ as its subgraph. We have $\chi(W_n) = \chi(S'(W_n)) = 3$ or $\chi(W_n) = \chi(S'(W_n)) = 4$ depending upon $n$ even or odd.

Thus, $\chi[S'(W_n)] = \begin{cases} 3, & n \text{ even} \\ 4, & n \text{ odd} \end{cases}$

Theorem 4.3.10.

\[ \varphi[S'(W_n)] = \begin{cases} 4, & n = 3 \\ 3, & n = 4 \\ 5, & n = 5, 6, 8 \\ 6, & n = 7 \\ 6, & n \geq 9 \end{cases} \]

Proof. To prove the result we continue with the terminology and notations used in Lemma 4.3.9 and consider the following cases.

Case 1: $n = 3$

In this case the graph $S'(W_3)$ contains an odd cycle. Then by Proposition 3.3.3, $\chi[S'(W_3)] \geq 3$. As $m[S'(W_3)] = 4$ and by Lemma 4.3.9, $\chi[S'(W_3)] = 4$. We have $4 \leq \varphi[S'(W_3)] \leq 4$ by Proposition 3.3.4. Thus, $\varphi[S'(W_3)] = 4$.

Case 2: $n = 4$

In this case the graph $S'(W_4)$ contains an odd cycle. Then by Proposition 3.3.3, $\chi[S'(W_4)] \geq 3$. As $m[S'(W_4)] = 5$ and by Lemma 4.3.9, $\chi[S'(W_4)] = 3$. Then by Proposition 3.3.4, we have $3 \leq \varphi[S'(W_4)] \leq 5$.

If $\varphi[S'(W_4)] = 5$ then by Proposition 3.3.2, the graph $S'(W_4)$ must have five vertices of degree at least 4 which is possible. But due to the adjacency of vertices of the graph $S'(W_4)$ any proper coloring with five colors have at least one color class which does not have color dominating vertices hence it will not be b-coloring for the graph $S'(W_4)$. Thus, $\varphi[S'(W_4)] \neq 5$. 

\[ \varphi[S'(W_n)] = \begin{cases} 4, & n = 3 \\ 3, & n = 4 \\ 5, & n = 5, 6, 8 \\ 6, & n = 7 \\ 6, & n \geq 9 \end{cases} \]
Suppose $\phi[S'(W_4)] = 4$. Now consider the color class $c = \{1, 2, 3, 4\}$ and define the color function as $f : V[S'(W_4)] \rightarrow \{1, 2, 3, 4\}$ as $f(v) = 4 = f(v')$, $f(v_1) = 1$, $f(v_2) = 2$, $f(v_1') = 1$, $f(v_2') = 2$, $f(v_3) = 3$, $f(v_4) = 3$ which in turn forces us to assign $f(v_3) = 1$, $f(v_4) = 2$. This proper coloring gives the color dominating vertices for color classes 1, 2 and 4 but not for 3 which contradicts our assumption. Thus, $\phi[S'(W_4)] \neq 4$. Hence, we can color the graph by three colors. For b-coloring, consider the color class $c = \{1, 2, 3\}$ and define the color function as $f : V[S'(W_4)] \rightarrow \{1, 2, 3\}$ as $f(v_1) = 1 = f(v_1'), f(v_2) = 2 = f(v_2'), f(v_3) = 1 = f(v_3'), f(v_4) = 2 = f(v_4'), f(v) = 3 = f(v')$. This proper coloring gives the color dominating vertices as $cdv(1) = v_1$, $cdv(2) = v_2$, $cdv(3) = v$. Thus $\phi[S'(W_4)] = 3$.

**Case 3: $n = 5, 6, 8$**

**Subcase 1: $n = 5$**

In this case the graph $S'(W_5)$ contains an odd cycle. Then by Proposition 3.3.3,

$\chi[S'(W_5)] \geq 3$. As $m[S'(W_5)] = 6$ and by Lemma 4.3.9, $\chi[S'(W_5)] = 4$. Then by Proposition 3.3.4, we have $4 \leq \phi(S'(W_5)) \leq 6$.

If $\phi(S'(W_5)) = 6$ then by Proposition 3.3.2, the graph $S'(W_5)$ must have six vertices of degree at least five which is possible. But due to the adjacency of vertices of the graph $S'(W_5)$ any proper coloring with six colors have at least one color class which does not have color dominating vertices. Hence it will not be b-coloring for the graph $S'(W_5)$.

Thus, $\phi(S'(W_5)) \neq 6$.

Suppose $\phi(S'(W_5)) = 5$. Now consider the color class $c = \{1, 2, 3, 4, 5\}$ and define the color function as $f : V[S'(W_5)] \rightarrow \{1, 2, 3, 4, 5\}$ as $f(v) = 5 = f(v'), f(v_1) = 3, f(v_2) = 1, f(v_3) = 2, f(v_4) = 3, f(v_5) = 4, f(v_1') = 2, f(v_2') = 4, f(v_3') = 4, f(v_4') = 1, f(v_5') = 1$.

This proper coloring gives the color dominating vertices as $cdv(1) = v_2$, $cdv(2) = v_3$, $cdv(3) = v_4$, $cdv(4) = v_5$, $cdv(5) = v$. Thus, $\phi(S'(W_5)) = 5$.

**Subcase 2: $n = 6, 8$**

In this case the graph $S'(W_n)$ contains an odd cycle. Then by Proposition 3.3.3,

$\chi[S'(W_n)] \geq 3$. As $m[S'(W_n)] = 7$ and by Lemma 4.3.9, $\chi[S'(W_n)] = 3$. Then by Proposition 3.3.4, we have $3 \leq \phi(S'(W_n)) \leq 7$.

If $\phi[S'(W_n)] = 7$ then by Proposition 3.3.2, the graph $S'(W_n)$ must have seven vertices
of degree at least six which is possible. But due to the adjacency of the vertices of graph $S'(W_n)$ any proper coloring with seven colors have at least one color class which does not have color dominating vertices. Hence it will not be b-coloring for the graph $S'(W_n)$. Thus, $\varphi[S'(W_n)] \neq 7$.

Suppose $\varphi[S'(W_n)] = 6$. Now consider the color class $c = \{1, 2, 3, 4, 5, 6\}$ and define the color function as $f: V[S'(W_n)] \to \{1, 2, 3, 4, 5, 6\}$ as $f(v) = 6 = f(v'), f(v_1) = 3, f(v_2) = 1, f(v_3) = 2, f(v_4) = 3, f(v_5) = 4, f(v'_1) = 4, f(v'_2) = 4, f(v'_3) = 5, f(v'_4) = 5, f(v'_5) = 1$ which in turn forces us to assign $f(v_6) = 2, f(v'_6) = 1$. This proper coloring gives the color dominating vertices for color classes 1, 2, 3, 4 and 6 but not for 5 which contradicts our assumption. Thus, $\varphi[S'(W_n)] \neq 6$.

Suppose that $S'(W_n)$ has b-coloring with 5 colors. Now consider the color class $c = \{1, 2, 3, 4, 5\}$ and define the color function as $f: V[S'(W_n)] \to \{1, 2, 3, 4, 5\}$ as $f(v) = 5 = f(v'), f(v_1) = 3, f(v_2) = 1, f(v_3) = 2 = f(v'_3), f(v_4) = 3 = f(v'_4), f(v_5) = 4, f(v_6) = 2, f(v'_1) = 4, f(v'_2) = 4, f(v'_5) = 1, f(v'_6) = 1$. This proper coloring gives the color dominating vertices as $cdv(1) = v_2, cdv(2) = v_3, cdv(3) = v_4, cdv(4) = v_5, cdv(5) = v$. Thus, $\varphi[S'(W_n)] = 5$.

**Case 4: n = 7**

In this case the graph $S'(W_7)$ contains an odd cycle. Then by Proposition 3.3.3, $\chi[S'(W_7)] \geq 3$. As $m[S'(W_7)] = 7$ and by Lemma 4.3.9, $\chi[S'(W_7)] = 4$. Then by Proposition 3.3.4, we have $4 \leq \varphi[S'(W_7)] \leq 7$.

Suppose $\varphi[S'(W_7)] = 7$. Now consider the color class $c = \{1, 2, 3, 4, 5, 6, 7\}$ and define the color function as $f: V[S'(W_7)] \to \{1, 2, 3, 4, 5, 6, 7\}$ as $f(v) = 7, f(v') = 6, f(v_1) = 5, f(v_2) = 1, f(v_3) = 2, f(v_4) = 3, f(v_5) = 1, f(v_6) = 4, f(v'_1) = 1, f(v'_2) = 4, f(v'_3) = 4, f(v'_4) = 5, f(v'_5) = 5, f(v'_6) = 4$ which in turn forces us to assign $f(v_7) = 2, f(v'_7) = 3$. This proper coloring gives the color dominating vertices for color classes 1, 2, 3, 4 and 5 but not for 6 and 7 which contradicts our assumption. Thus, $\varphi[S'(W_7)] \neq 7$.

Suppose that $S'(W_7)$ has b-coloring with 6 colors. Now consider the color class $c = \{1, 2, 3, 4, 5, 6\}$ and define the color function $f: V[S'(W_7)] \to \{1, 2, 3, 4, 5, 6\}$ as $f(v) = 6 = f(v'), f(v_1) = 3, f(v_2) = 1, f(v_3) = 2, f(v_4) = 3, f(v_5) = 4, f(v_6) = 2, f(v_7) = 5, f(v'_1) = 4, f(v'_2) = 4, f(v'_3) = 5, f(v'_4) = 5, f(v'_5) = 1, f(v'_6) = 1, f(v'_7) = 5$. This
proper coloring gives the color dominating vertices as \( cdv(1) = v_2, cdv(2) = v_3, cdv(3) = v_4, cdv(4) = v_5, cdv(5) = v_7, cdv(6) = v \). Thus, \( \varphi[S'(W_7)] = 6 \).

**Case 5: \( n \geq 9 \)**

For \( n = 9 \), the graph \( S'(W_9) \) contains an odd cycle. Then by Proposition 3.3.3, \( \chi[S'(W_9)] \geq 3 \). As \( m[S'(W_9)] = 7 \) and by Lemma 4.3.9, \( \chi[S'(W_9)] = 4 \). Then by Proposition 3.3.4, we have \( 4 \leq \varphi[S'(W_9)] \leq 7 \).

Suppose \( \varphi[S'(W_9)] = 7 \). Now consider the color class \( c = \{1, 2, 3, 4, 5, 6, 7\} \) and define the color function \( f : V[S'(W_9)] \to \{1, 2, 3, 4, 5, 6, 7\} \) as \( f(v) = 6, f(v') = 7, f(v_1) = 3, f(v_2) = 1, f(v_3) = 2, f(v_4) = 3, f(v_5) = 4, f(v_6) = 1, f(v'_1) = 4, f(v'_2) = 4, f(v'_3) = 5, f(v'_4) = 5, f(v'_5) = 1, f(v'_6) = 2, f(v'_7) = 5, f(v'_8) = 5, f(v_8) = 3, f(v_9) = 4 \) which in turn forces us to assign \( f(v_9) = 2 = f(v'_9) \). This proper coloring gives the color dominating vertices for color classes 1, 2, 3, 4 and 5 but not for 6 and 7 which contradicts our assumption. Thus, \( \varphi[S'(W_9)] \neq 7 \).

Suppose that \( S'(W_9) \) has b-coloring with 6 colors. Now consider the color class \( c = \{1, 2, 3, 4, 5, 6\} \) and define the color function \( f : V[S'(W_9)] \to \{1, 2, 3, 4, 5, 6\} \) as \( f(v) = 6 = f(v'), f(v_1) = 3, f(v_2) = 1, f(v_3) = 2, f(v_4) = 3, f(v_5) = 4, f(v_6) = 2, f(v_7) = 5, f(v_8) = 3, f(v'_1) = 4, f(v'_2) = 4, f(v'_3) = 5, f(v'_4) = 5, f(v'_5) = 1, f(v'_6) = 1, f(v'_7) = 5, f(v'_8) = 4 \). This proper coloring gives the color dominating vertices as \( cdv(1) = v_2, cdv(2) = v_3, cdv(3) = v_4, cdv(4) = v_5, cdv(5) = v_7, cdv(6) = v \). Thus, \( \varphi[S'(W_9)] = 6 \).

**For \( n \geq 9 \)**

We repeat the colors as in the above graph \( S'(W_9) \) for the vertices \( \{v_1, v_2, \ldots, v_9, v'_1, v'_2, \ldots, v'_9, v, v'\} \) and for the remaining vertices assign the colors as \( f(v) = 6 = f(v'), f(v_{3k+7}) = 1 = f(v'_{3k+7}), f(v_{3k+8}) = 2 = f(v'_{3k+8}) \) where \( k \in \mathbb{N} \). Hence, \( \varphi[S'(W_9)] = 6 \), for all \( n \geq 9 \).

\[
\varphi[S'(W_n)] = \begin{cases} 
4, & n = 3 \\
3, & n = 4 \\
5, & n = 5, 6, 8 \\
6, & n = 7 \\
6, & n \geq 9 
\end{cases}
\]

Hence, \( \varphi[S'(W_n)] \).
Illustration 4.3.11. Splitting graph of wheel $W_5$ and its b-coloring using five colors is shown in Figure 4.9.

**Theorem 4.3.12.** $S'(W_n)$ is b-continuous.

**Proof.** To prove the result we continue with the terminology and notations used in Lemma 4.3.9 and consider the following cases.

**Case 1:** $n = 3$

In this case the graph $S'(W_3)$ is b-continuous as $\chi[S'(W_3)] = \phi[S'(W_3)] = 4$.

**Case 2:** $n = 4$

In this case the graph $S'(W_4)$ is b-continuous as $\chi[S'(W_4)] = \phi[S'(W_4)] = 3$.

**Case 3:** $n = 5$

In this case by Lemma 4.3.9, $\chi[S'(W_5)] = 4$ and by Theorem 4.3.10, $\phi[S'(W_5)] = 5$. Hence, b-coloring exists for every integer satisfying $\chi[S'(W_5)] \leq K \leq \phi[S'(W_5)]$ (Here $K = 4, 5$). Thus, $S'(W_5)$ is b-continuous.

**Case 4:** $n = 6$

In this case by Lemma 4.3.9, $\chi[S'(W_6)] = 3$ and by Theorem 4.3.10, $\phi[S'(W_6)] = 5$. It is obvious that b-coloring for the graph $S'(W_6)$ is possible using the number of colors $K = 3, 5$. Now for $K = 4$ the b-coloring for the graph $S'(W_6)$ is as follows.
Consider the color class \( c = \{1, 2, 3, 4\} \) and define the color function \( f : V[S'(W_6)] \rightarrow \{1, 2, 3, 4\} \) as \( f(v) = f(v') = 4, f(v_1) = f(v'_1) = 3, f(v_2) = f(v'_2) = 1, f(v_3) = f(v'_3) = 2, f(v_4) = f(v'_4) = 3, f(v_5) = f(v'_5) = 1, f(v_6) = f(v'_6) = 2 \). This proper coloring gives the color dominating vertices as \( cdv(1) = v_2, cdv(2) = v_3, cdv(3) = v_4, cdv(4) = v \). Thus, \( S'(W_6) \) is four colorable. Hence \( b \)-coloring exists for every integer \( K \) satisfying \( \chi[S'(W_6)] \leq K \leq \varphi[S'(W_6)] \)(Here \( K = 3, 4, 5 \)). Consequently \( S'(W_6) \) is \( b \)-continuous.

**Case 5: \( n = 7 \)**

By Lemma 4.3.9, \( \chi[S'(W_7)] = 4 \) and by Theorem 4.3.10, \( \varphi[S'(W_7)] = 6 \). It is obvious that \( b \)-coloring for the graph \( S'(W_7) \) is possible using the number of colors \( K = 4, 6 \).

Now for \( K = 5 \) the \( b \)-coloring for the graph \( S'(W_7) \) is as follows.

Consider the color class \( c = \{1, 2, 3, 4, 5\} \) and define the color function \( f : V[S'(W_7)] \rightarrow \{1, 2, 3, 4, 5\} \) as \( f(v) = f(v') = 5, f(v_1) = f(v'_1) = 3, f(v_2) = f(v'_2) = 4, f(v_3) = f(v'_3) = 2, f(v_4) = f(v'_4) = 3, f(v_5) = f(v'_5) = 1, f(v_6) = f(v'_6) = 2, f(v_7) = 1 = f(v'_7) \). This proper coloring gives the color dominating vertices as \( cdv(1) = v_2, cdv(2) = v_3, cdv(3) = v_4, cdv(4) = v_5, cdv(5) = v \). Thus, \( S'(W_7) \) is five colorable. Hence, \( b \)-coloring exists for every integer \( K \) satisfying \( \chi[S'(W_7)] \leq K \leq \varphi[S'(W_7)] \)(Here \( K = 4, 5, 6 \)). Hence \( S'(W_7) \) is \( b \)-continuous.

**Case 6: \( n = 8 \)**

By Lemma 4.3.9, \( \chi[S'(W_8)] = 3 \) and by Theorem 4.3.10, \( \varphi[S'(W_8)] = 5 \). It is obvious that \( b \)-coloring for the graph \( S'(W_8) \) is possible using the number of colors \( K = 3, 5 \).

Now for \( K = 4 \) the \( b \)-coloring for the graph \( S'(W_8) \) is as follows.

Consider the color class \( c = \{1, 2, 3, 4\} \) and define the color function as \( f : V[S'(W_8)] \rightarrow \{1, 2, 3, 4, 5\} \) as \( f(v) = f(v') = 4, f(v_1) = f(v'_1) = 3, f(v_2) = f(v'_2) = 1, f(v_3) = f(v'_3) = 2 = f(v'_4), f(v_4) = 3 = f(v'_4), f(v_5) = 1 = f(v'_5), f(v_6) = 2 = f(v'_6), f(v_7) = 1 = f(v'_7), f(v_8) = 2 = f(v'_8) \). This proper coloring gives the color dominating vertices as \( cdv(1) = v_2, cdv(2) = v_3, cdv(3) = v_4, cdv(4) = v \). Thus, \( S'(W_8) \) is four colorable. Hence, \( b \)-coloring exists for every integer \( K \) satisfying \( \chi[S'(W_8)] \leq K \leq \varphi[S'(W_8)] \)(Here \( K = 3, 4, 5 \)). Thus, \( S'(W_8) \) is \( b \)-continuous.
Case 7: \( n \geq 9 \)

**For \( n = 9 \)**

By Lemma 4.3.9, \( \chi[S'(W_9)] = 4 \) and by Theorem 4.3.10, \( \varphi[S'(W_9)] = 6 \). It is obvious that b-coloring for the graph \( S'(W_9) \) is possible using the number of colors \( K = 4, 6 \).

Now for \( K = 5 \) the b-coloring for the graph \( S'(W_9) \) is as follows.

Consider the color class \( c = \{1, 2, 3, 4, 5\} \) and define the color function as \( f : V[S'(W_9)] \rightarrow \{1, 2, 3, 4, 5\} \) as \( f(v) = f(v') = 5, f(v_1) = 3, f(v'_1) = 4, f(v_2) = 1, f(v'_2) = 4, f(v_3) = 2, f(v'_3) = 2, f(v_4) = 3, f(v'_4) = 1, f(v_5) = 4, f(v'_5) = 1, f(v_6) = 2, f(v'_6) = 2, f(v_7) = 1 = f(v'_7), f(v_8) = 2, f(v'_8) = 2, f(v_9) = f(v'_9) = 1 \). This proper coloring gives the color dominating vertices as \( cdv(1) = v_2, cdv(2) = v_3, cdv(3) = v_4, cdv(4) = v_5, cdv(5) = v \).

Thus, \( S'(W_9) \) is five colorable. Hence, b-coloring exists for every integer \( K \) satisfying \( \chi[S'(W_9)] \leq K \leq \varphi[S'(W_9)] \) (Here \( K = 4, 5, 6 \)). Hence, \( S'(W_9) \) is b-continuous.

**For odd \( n \geq 9 \)**

In this case repeat the colors as in \( S'(W_9) \) for the vertices \( \{v_1, v_2, \ldots, v_9, v'_1, v'_2, \ldots, v'_9, v, v'\} \) and for the remaining vertices assign the colors as follows

**When \( K = 5 \)**

\( f(v') = f(v) = 5, f(v_{3k+7}) = f(v'_{3k+7}) = 1, f(v_{3k+8}) = f(v'_{3k+8}) = 2, k \in N \)

**For even \( n \geq 9 \)**

In this case we repeat the color assignment as in case \( n = 8 \) discussed above for the vertices \( \{v, v', v_1, v_2, \ldots, v_8, v'_1, v'_2, \ldots, v'_8\} \) and for remaining vertices gives the colors as follows.

**When \( K = 4 \)**

\( f(v') = f(v) = 4, f(v_{2k+7}) = 1 = f(v'_{2k+7}), f(v_{2k+8}) = 2 = f(v'_{2k+8}). k \in N. \)

**When \( K = 5 \)**

\( f(v') = f(v) = 5, f(v_{2k+8}) = 1 = f(v'_{2k+8}), f(v_{2k+9}) = 2 = f(v'_{2k+9}). k \in N. \)

Therefore, \( S'(W_n) \) is b-continuous.
Illustration 4.3.13. For the graph $S'(W_6)$, $\chi(S'(W_6)) = 3$ and $\phi(S'(W_6)) = 3$. According to the definition of b-continuity $\chi(S'(W_6)) \leq K \leq \phi(S'(W_6)) \Rightarrow 3 \leq K \leq 5$. The various b-colorings using three, four and five colors are shown in Figure 4.10.

In accordance with the graph $S'(W_n)$ considered in above Theorem 4.3.10 we state the following corollary.
Corollary 4.3.14.

\[
S_b[S'(W_n)] = \begin{cases} 
\{4\}, & n = 3 \\
\{3\}, & n = 4 \\
\{4, 5\}, & n = 5 \\
\{3, 4, 5\}, & n = 6, 8 \\
\{4, 5, 6\}, & n = 7 \\
\{4, 5, 6\} & \text{for odd } n \geq 9 \\
\{3, 4, 5\} & \text{for even } n > 9.
\end{cases}
\]

**Proof.** The proof is straightforward. \(\blacksquare\)

### 4.4 Duplication of rim edges in wheel \(W_n\) and \(b\)-coloring

**Lemma 4.4.1.** Let \(G\) be the graph obtained by duplication of rim edges altogether of wheel \(W_n\), then

\[
\chi(G) = \begin{cases} 
3, & n \text{ even} \\
4, & n \text{ odd}.
\end{cases}
\]

**Proof.** Let \(v_1, v_2, \ldots, v_n\) be the rim vertices, \(v\) be the apex of \(W_n\) while \(e_1, e_2, \ldots, e_n\) be the rim edges of \(W_n\) which are to be duplicated by the edges \(e'_1, e'_2, \ldots, e'_n\) respectively. Then the resultant graph \(G\) has \(|V(G)| = 3n + 1\) and \(|E(G)| = 7n\).

**Case 1:** \(n\) is even.

In this case \(G\) contains even \(W_n\) as an induced subgraph. Since \(\chi(W_n) = 3 \Rightarrow \chi(G) = 3\).

**Case 2:** \(n\) is odd.

In this case \(G\) contains odd \(W_n\) as an induced subgraph. Since \(\chi(W_n) = 4 \Rightarrow \chi(G) = 4\).

\(\blacksquare\)

We continue with the terminology and notations used in Lemma 4.4.1 to prove the next two results.
Theorem 4.4.2.

\[ \varphi(G) = \begin{cases} 
4, & n = 3 \\
5, & n = 4 \\
6, & n = 5 \\
6, & n \geq 6.
\end{cases} \]

**Proof.** To prove the result consider the following cases.

**Case 1:** \( n = 3 \)

By Lemma 4.4.1, \( \chi(G) = 4 \) and also \( m(G) = 4 \). Then by Proposition 3.3.4, it follows that \( \chi(G) \leq \varphi(G) \leq m(G) \Rightarrow 4 \leq \varphi(G) \leq 4 \). Thus, \( \varphi(G) = 4 \).

**Case 2:** \( n = 4 \)

By Lemma 4.4.1, \( \chi(G) = 3 \) and also \( m(G) = 5 \). Then by Proposition 3.3.4, \( \chi(G) \leq \varphi(G) \leq m(G) \Rightarrow 3 \leq \varphi(G) \leq 5 \).

If \( \varphi(G) = 5 \) then the graph \( G \) must have five vertices of degree at least four which is possible according to Proposition 3.3.2. We consider the color class \( c = \{1, 2, 3, 4, 5\} \).

To assign the proper coloring to the vertices define the color function \( f : V(G) \to \{1, 2, 3, 4, 5\} \) as \( f(v) = 5, f(v_1) = 1, f(v_2) = 2, f(v_3) = 3, f(v_4) = 4, f(v'_1) = 1, f(v'_2) = 2, f(v'_3) = 2, f(v'_4) = 2, f(v''_1) = 3, f(v''_2) = 3, f(v''_3) = 4, f(v''_4) = 1 \). This proper coloring gives the color dominating vertices as \( cdv(1) = v_1, cdv(2) = v_2, cdv(3) = v_3, cdv(4) = v_4, cdv(5) = v \). Thus, \( \varphi(G) = 5 \).

**Case 3:** \( n = 5 \)

By Lemma 4.4.1, \( \chi(G) = 4 \) and also \( m(G) = 6 \). Then by Proposition 3.3.4, \( \chi(G) \leq \varphi(G) \leq m(G) \Rightarrow 4 \leq \varphi(G) \leq 6 \).

Suppose \( \varphi(G) = 6 \) then according to the Proposition 3.3.2, the graph \( G \) must have six vertices of degree at least five which is possible. Consider the color class \( c = \{1, 2, 3, 4, 5, 6\} \). To assign the proper coloring to the vertices define the color function \( f : V(G) \to \{1, 2, 3, 4, 5\} \) as \( f(v) = 6, f(v_1) = 5, f(v_2) = 1, f(v_3) = 2, f(v_4) = 3, f(v_5) = 4, f(v'_1) = 2, f(v'_2) = 4, f(v'_3) = 1, f(v'_4) = 1, f(v''_1) = 2, f(v''_2) = 3, f(v''_3) = 3, f(v''_4) = 2, f(v''_5) = 5 \). This proper coloring gives the color dominating vertices as \( cdv(1) = v_2, cdv(2) = v_3, cdv(3) = v_4, cdv(4) = v_5, cdv(5) = v_1, cdv(6) = v \). Thus,
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\(\varphi(G) = 6.\)

**Case 4: \(n \geq 6\)**

By Lemma 4.4.1, \(\chi(G) = 3\) and also \(m(G) = 6.\) Then by Proposition 3.3.4, \(\chi(G) \leq \varphi(G) \leq m(G) \Rightarrow 3 \leq \varphi(G) \leq 6.\)

Suppose \(\varphi(G) = 6\) then according to Proposition 3.3.2, the graph \(G\) must have six vertices of degree at least five which is possible. Consider the color class \(c = \{1, 2, 3, 4, 5, 6\}\).

To assign the proper coloring to the vertices define the color function \(f : V(G) \rightarrow \{1, 2, 3, 4, 5, 6\}\) as \(f(v) = 6, f(v_1) = 2, f(v_2) = 5, f(v_3) = 1, f(v_4) = 2, f(v_5) = 3, f(v_6) = 4, f(v'_1) = 3, f(v'_2) = 3, f(v'_3) = 4, f(v'_4) = 4, f(v'_5) = 1, f(v'_6) = 5, f(v''_1) = 3, f(v''_2) = 3, f(v''_3) = 4, f(v''_4) = 1, f(v''_5) = 5, f(v''_6) = 5.\) This proper coloring gives the color dominating vertices as \(\text{cdv}(1) = v_3, \text{cdv}(2) = v_4, \text{cdv}(3) = v_5, \text{cdv}(4) = v_6, \text{cdv}(5) = v_2, \text{cdv}(6) = v.\) Thus, \(\varphi(G) = 6.\)

**For \(n \geq 6\)**

We repeat the colors as in the above graph \(G\) for the vertices \(\{v_1, v_2, \ldots, v_6, v'_1, v'_2, \ldots, v'_6, v''_1, v''_2, \ldots, v''_6, v\}\) and for the remaining vertices assign the colors as follows.

**When \(n\) is even** \(f(v) = 6, f(v_{3k+5}) = 3, f(v'_{3k+5}) = f(v''_{3k+5}) = 4, k \in \mathbb{N}.\)

**When \(n\) is odd** \(f(v) = 6, f(v_{3k+4}) = 5, f(v'_{3k+4}) = f(v''_{3k+4}) = 2, k \in \mathbb{N}.\)

Hence, \(\varphi(G) = 6, n \geq 6.\)

Thus, \(\varphi(G) = \begin{cases} 4, & n = 3 \\ 5, & n = 4 \\ 6, & n = 5 \\ 6, & n \geq 6. \end{cases} \)
Illustration 4.4.3. A b-coloring of duplication of rim edges of wheel $W_5$ is shown in Figure 4.11.

**Figure 4.11:** Duplication of rim edges of wheel $W_5$ and its b-coloring

**Theorem 4.4.4.** $G$ is b-continuous.

**Proof.** To prove the result consider the following cases.

**Case 1:** $n = 3$

In this case the graph $G$ is b-continuous as $\chi(G) = \varphi(G) = 4$.

**Case 2:** $n = 4$

In this case by Lemma 4.4.1, $\chi(G) = 3$ and by Theorem 4.4.2, $\varphi(G) = 5$. It is obvious that b-coloring for the graph $G$ is possible using the number of colors $K = 3, 5$. Does b-coloring exists for $K = 4$? To answer this

Consider the color class $c = \{1, 2, 3, 4\}$ and assign the proper coloring to the vertices, define the color function $f: V(G) \rightarrow \{1, 2, 3, 4\}$ as $f(v) = 4, f(v_1) = 1, f(v_2) = 2, f(v_3) = 3, f(v_4) = 2, f(v_1') = 1, f(v_2') = 2, f(v_3') = 1, f(v_4') = 2, f(v_1'') = 3, f(v_2'') = 3, f(v_3'') = 3, f(v_4'') = 1$. This proper coloring gives the color dominating vertices as $cdv(1) = v_1, cdv(2) = v_2, cdv(3) = v_3, cdv(4) = v$. Thus graph $G$ is four colorable. Hence b-coloring exists for every integer $K$ satisfying $\chi(G) \leq K \leq \varphi(G)$ (Here $K = 3, 4, 5$). Consequently, $G$ is b-continuous.
Case 3: \( n = 5 \)

By Lemma 4.4.1, \( \chi(G) = 4 \) and by Theorem 4.4.2, \( \varphi(G) = 6 \). It is obvious that b-coloring for the graph \( G \) is possible using the number of colors \( k = 4, 6 \). To establish the b-continuity we will show that the b-coloring exists for \( K = 5 \).

For that consider the color class \( c = \{1, 2, 3, 4, 5\} \). To assign the proper coloring to the vertices define the color function \( f : V(G) \to \{1, 2, 3, 4, 5\} \) as \( f(v) = 5, f(v_1) = 2, f(v_2) = 1, f(v_3) = 2, f(v_4) = 3, f(v_5) = 4, f(v'_1) = 2 = f(v''_1), f(v'_2) = 4 = f(v''_2), f(v'_3) = 3, f(v''_3) = 2, f(v'_4) = f(v''_4) = 1, f(v'_5) = 3, f(v''_5) = 1 \). This proper coloring gives the color dominating vertices as \( cdv(1) = v_2, cdv(2) = v_3, cdv(3) = v_4, cdv(4) = v_5, cdv(5) = v \). Thus \( G \) is five colorable. Hence b-coloring exists for every integer \( K \) satisfying \( \chi(G) \leq K \leq \varphi(G) \). Thus, \( G \) is b-continuous.

Case 4: \( n \geq 6 \)

For \( n = 6 \), By Lemma 4.4.1, \( \chi(G) = 3 \) and by Theorem 4.4.2, \( \varphi(G) = 6 \). It is obvious that b-coloring for the graph \( G \) is possible using the number of colors \( K = 3, 6 \).

When \( K = 4 \) consider the color class \( c = \{1, 2, 3, 4\} \). To assign the proper coloring to the vertices as \( f(v) = 4, f(v_1) = f(v'_1) = f(v''_1) = 1, f(v_2) = f(v'_2) = f(v''_2) = 2, f(v_3) = f(v'_3) = f(v''_3) = 3, f(v_4) = f(v'_4) = f(v''_4) = 1, f(v_5) = f(v'_5) = f(v''_5) = 2, f(v_6) = f(v'_6) = f(v''_6) = 3 \). This proper coloring gives the color dominating vertices as \( cdv(1) = v_1, cdv(2) = v_2, cdv(3) = v_3, cdv(4) = v \). Thus, \( G \) is four colorable.

For \( K = 5 \) the b-coloring for the graph \( G \) is as follows. Consider the color class \( c = \{1, 2, 3, 4, 5\} \). To assign the proper coloring to the vertices define the color function \( f : V(G) \to \{1, 2, 3, 4, 5\} \) as \( f(v) = 5, f(v_1) = 1, f(v_2) = 2, f(v_3) = 3, f(v_4) = 4, f(v_5) = 1, f(v'_6) = 3, f(v''_6) = 4, f(v'_2) = 2, f(v''_2) = 2, f(v'_4) = 4, f(v''_4) = 4, f(v'_5) = 1, f(v''_5) = 2 \). This proper coloring gives the color dominating vertices as \( cdv(1) = v_2, cdv(2) = v_3, cdv(3) = v_4, cdv(4) = v_5, cdv(5) = v \). Thus, \( G \) is 5 colorable. Hence b-coloring exists for every integer \( K \) satisfying \( \chi(G) \leq K \leq \varphi(G) \). Thus, \( G \) is b-continuous.

When \( n > 6 \)

In this case assign same colors for the vertices \( \{v_1, v_2, \ldots, v_6, v'_1, v'_2, \ldots, v'_6, v''_1, v''_2, \ldots, v''_6\} \) and for the remaining vertices assign the proper coloring using colors from the set
\{1, 2, 3, 4, 5, 6\}. Thus, $G$ is $b$-continuous. 

**Illustration 4.4.5.** If the graph $G$ is obtained from duplication of rim edges in $W_5$ then $\chi(G) = 4$ and $\varphi(G) = 6$. According to the definition of $b$-continuity $\chi(G) \leq K \leq \varphi(G) \Rightarrow 4 \leq K \leq 6$. The various $b$-colorings using four, five and six colors are shown in Figure 4.12.

![Figure 4.12: Duplication of rim edges of wheel $W_5$ and its b-colorings](image)

In accordance with the graph $G$ considered in above Theorem 4.4.2 we state the following corollary.

**Corollary 4.4.6.**

$$S_b(G) = \begin{cases} 
\{4\} & n = 3 \\
\{3, 4, 5\} & n = 4 \\
\{4, 5, 6\} & n = 5 \\
\{3, 4, 5, 6\} & \text{for even } n \geq 6 \\
\{4, 5, 6\} & \text{for odd } n > 6
\end{cases}$$
Proof. The proof is straightforward.

4.5 Concluding remarks

The clustering problems occurring in the process of data mining, web-services classifications and decomposition of large systems can be handled with b-coloring of graphs. In chapter 3 we have investigated b-chromatic number of some cycle and path related graphs while the present chapter was intended to investigate the b-chromatic number of some wheel related graphs.

4.6 Some open problems

- To determine the bounds on b-chromatic number and to identify the graph families achieving such bounds.

- To prove some characterizations for b-coloring of graph.

- To investigate b-chromatic number for a graph obtained from the given graph by means of some graph operations.

- To identify the relation between b-coloring and other variants of proper coloring of graphs.

The next chapter is aimed to give a brief account of one more variant of proper coloring.