REFERENCES


APPENDIX

In this appendix, we shall survey those definitions and results which are subsequently used throughout the thesis. We shall take the definitions and results from [1], [2], and [3]. A collection \( V \) of elements \( \phi, \psi, \theta, \ldots \) is called a linear-space if the following axioms are satisfied.

1) There is an operation +, called "addition" by which pair of elements \( \phi \) and \( \psi \) can be combined to yield an element \( \phi + \psi \) in \( V \) such a way that the following properties are satisfied.

   (1.a) \( \phi + \psi = \psi + \phi \) (commutativity)

   (1.b) \( (\phi + \psi) + \sigma = \phi + (\psi + \sigma) \) (associativity)

   (1.c) There exists a unique element 0, called zero in \( V \), such that \( \phi + 0 = \phi \) for every \( \phi \in V \).

   (1.d) For every \( \phi \in V \), there exists a unique element \( \phi \) in \( V \) such that \( \phi + (-\phi) = 0 \).

2) There is an operation, called "multiplication by a complex number" by which any complex number \( \alpha \) and any \( \phi \in V \) can be combined to yield an element \( \alpha \phi \in V \) in such a way that the following conditions are fulfilled.
(2.a) \( \alpha \beta \phi = (\alpha \beta) \phi \) for all complex numbers \( \alpha \) and \( \beta \)

(2.b) \( 1 \cdot \phi = \phi \) (1 denotes the number one)

3) The following distributive laws hold:

(3.a) \( \alpha(\phi + \psi) = \alpha \phi + \alpha \psi \)

(\( \alpha + \beta \)) \( \phi = \alpha \phi + \beta \phi \).

[\( A.2 \)] A subset \( U \) of a linear space \( V \) is called a subspace of \( V \) if for every \( \phi, \psi \in U \) then \( \alpha \phi, \phi + \psi \) both are in \( V \).

[\( A.3 \)] Let \( n \) be a linear space. A seminorm on \( V \) is a rule \( p \) that assigns a real number \( p(\phi) \) to each \( \phi \in V \) and that satisfies the following axioms. Here \( \phi \) and \( \psi \) are arbitrary elements of \( V \) and \( \alpha \) is any complex number,

\[
(1) \quad p(\alpha \phi) = |\alpha| p(\phi)
\]

\[
(2) \quad p(\phi + \psi) \leq p(\phi) + p(\psi).
\]

[\( A.4 \)] Let \( S = \{ p_m \}_{m \in A} \) be a collection of seminorms on \( V \) where the \( m \) traverses a finite or infinite set \( A \). The collection \( S \) is said to be separating if for every \( \phi \neq 0 \) in \( V \) there is at least one \( p_m \) such that \( p_m(\phi) \neq 0 \).

In this case we call \( S \) a multinorm. A sufficient condition
for $S$ to be separating is that at least one of the $P_m$ be a norm.

(A-5) A Topological space is a set $V$ for which a collection $T$ of subsets of $V$ is specified and has the following properties.

(i) $V$ and the empty set are members of $T$.

(ii) Every union of members of $T$ is a member of $T$.

(iii) The intersection of any finite number of members of $T$ is a member of $T$.

The members of $T$ are called open sets and $T$ is said to be topology on $V$.

(A-6) A multinormed space $V$ is a linear space having a topology generated by a multinorm $S$. If $S$ is countable, $V$ is called a countably multinormed space.

(A-7) A sequence $\left\{ \phi_m \right\}_{m=1}^{\infty}$ is called convergent in $V$ if all $\phi_m$ are members of $V$ and if there exists a $\phi \in V$ such that for every neighbourhood $\sim \phi$ of $\phi$, $\phi_m$ is eventually in $\sim \phi$.

(A-8) Lemma: Let $V$ be a multinormed space with the multinorms $S$. A sequence $\left\{ \phi_m \right\}_{m=1}^{\infty}$ converges in $V$ to the limit $\phi$ if and only if for each $P \in S$, $P(\phi - \phi_m) \to 0$ as $m \to \infty$. The limit $\phi$ is unique.
Let $M$ be a subset of multinormed space $V$. $M$ is dense in $V$ if, for each $\phi \in V$, there exists a sequence
\[
\{\phi_m\}_{m=1}^{\infty}
\]
of elements in $M$ which converges in $V$ to $\phi$.

Let $V$ be a multinormed space. Then $\{\phi_m\}_{m=1}^{\infty}$ is a Cauchy sequence in $V$ if and only if all $\phi_m$ are in $V$, and for each $p \subseteq \mathbb{P}(\phi_m - \phi_n) \rightarrow 0$ as $m$ and $n$ tend to infinity independently.

**Sequence**

Every convergent is a Cauchy sequence but converse need not be true.

Every Cauchy sequence in $V$ is convergent, then $V$ is said to be complete. A complete countably multinormed space is called a Frechet space.

Let $V$ be a linear space and $T_1$ and $T_2$ denote two topologies generated respectively by two different multinorms $R = \bigcup_{m \in \mathcal{A}} \{P_m\}$ and $S = \bigcup_{m \in \mathcal{B}} \{P_m\}$. A necessary and sufficient condition for $T_1$ to be weaker than $T_2$ is that for each $p \in R$, there exist a finite set of seminorms $p_1, p_2, p_3, \ldots, p_n \in \mathcal{A}$ such that for every $\phi \in V$
\[
p'(\phi) \leq \max_{i} p_i(\phi), \ldots, p_n(\phi).
\]
or alternatively

\[ p'(\phi) < C \left[ p_1(\phi) + p_2(\phi) + \ldots + p_n(\phi) \right] \]

where \( C \) is a positive number. Both \( C \) and \( n \) depend on the choice of \( p' \).

[A-13] Let \( V \) be a multinormed space and \( U \) a linear subspace of \( V \). Also let \( \xi \) denote the multinorm for \( V \). Then \( \xi \) is also a multinorm on \( U \). The topology generated in \( U \) by \( \xi \) is called the topology induced on \( U \) by \( S \).

[A-14] Lemma: Let \( U \) and \( V \) be multinormed space and assume that \( U \) is a linear subspace of \( V \). Also, assume that the topology on \( U \) by \( V \) is stronger than the topology induced on \( U \) by \( V \). If \( \xi_{m/} \) converges to \( \phi \) in \( U \), then \( \phi_{m/} \) converges in \( V \) and has the same limit as it does in \( U \).

[A-15] Let \( V \) be a countably multinormed space. A rule that assigns a unique complex number to each \( \phi \in V \) is called a functional on \( V \). We shall denote this complex number by \( \langle f, \phi \rangle \). The collection of all continuous linear functionals \( f \) on the countably multinormed space \( V \) is called the dual of \( V \), and is denoted by \( V' \).
Theorem: Let $V$ be a linear space with a topology by the countable multinorm $\|p_m\|_{\infty}^{m=1}$ where $p_1$ is a norm. Let the countable multinorm $\|p'_m\|_{\infty}^{m=1}$ be defined by $p'_m = \max \{ p_1, p_2, p_3, \ldots, p_m \}$ for each continuous linear functional $f$ defined on $V$ there exist a positive constant $C$ and a non-negative integer $r$ such that for every $\varphi \in V$

$$|\langle f, \varphi \rangle| \leq C p_r(\varphi)$$

Here $C$ and $r$ depend on $f$ but not on $\varphi$.

Assume that $U$ and $V$ are countably multinormed spaces with $U$ being a subspace of $V$. By the restriction of $f \in V'$ to $U$ we mean that unique functional $g$ on $U$ defined by $\langle g, \varphi \rangle = \langle f, \varphi \rangle$ for every $\varphi \in U$.

Theorem: If $U$ and $V$ are countably multinormed spaces with $U$ being a dense subspace of $V$ and if the topology of $U$ is stronger than that induced on it by $V$, then $V'$ is a subspace of $U'$.

Theorem: Let $U$ and $V$ be countably multinormed spaces with $V$ being a dense subspace of $V$. Assume that the convergence of any sequence to zero in $U$ implies its convergence to zero in $V$. Then $V'$ is a subspace of $U'$.
If $V$ is a complete countably multinormed space, then its dual $V'$ is also complete.

Let $U$ and $V$ be countable union-spaces. Let $V = \bigcup_{m=1}^{\infty} V_m$ generated by the sequence of countably multinormed spaces. Assume that $U$ is a dense subspace of $V$ and the convergence concept for $U$ is stronger than the one for $V$. Then $V'$ is a subspace of $U'$.

Let $R$ be a linear mapping of the multinormed space $U$ into the multinormed space $V$, and let $S$ and $R$ be the multinorms for $U$ and $V$ respectively. A necessary and sufficient condition for $R$ to be continuous is that to every $p \in R$ the corresponding finite number of seminorms $p_1, p_2, p_3, \ldots, p_n \in S$ and a positive number $C$ such that for all $q \in U$ we have

$$p'(Rq) < C \max \left[ p_1(q), \ldots, p_n(q) \right]$$

or alternatively

$$p'(Rq) < C \left[ p_1(q) + p_2(q) + \ldots + p_n(q) \right]$$

The multinormed spaces $U$ and $V$ are called isomorphic, if there exists a one to one continuous linear
mapping $R$ of $U$ onto $V$ such that its inverse $R^{-1}$ is continuous linear mapping of $V$ into $U$. In this case $R$ is called on isomorphism from $U$ onto $V$, in addition $U = V$, then $R$ is called an automorphism on $U$.

[A-24] Assume that $U$ and $V$ are both countably multi-normed spaces and let $R$ be a continuous linear mapping of $U$ into $V$. We define the adjoint operator $R'$ on the dual space $V'$ by

$$\langle R' f, \delta \rangle = \langle f, R\delta \rangle,$$

where $f \in V'$ and $\delta$ traverses all of $U$.

[A-25] Theorem: If $U$ and $V$ are both countably multi-normed spaces or both countable union spaces and $R$ is a continuous linear mapping of $U$ into $V$ then the adjoint operator $R'$ is a continuous linear mapping of $V'$ into $U'$.

[A-26] If $U$ and $V$ are both countably multi-normed spaces or both countable-union-spaces and $R$ is a isomorphism from $U$ onto $V$ then $R'$ is an isomorphism from $V'$ onto $U'$. Moreover $(R'^{-1} = (R^{-1})$.

[A-27] $\mathbb{R}^n$ and $\mathbb{C}^n$ denote respectively the real and complex $n$ dimensional euclidean spaces. A compact set in $\mathbb{R}^n$ or $\mathbb{C}^n$ is simply a closed bounded set. If $I$ is an
open set in $\mathbb{R}^n$, if $K$ is a compact set in $\mathbb{R}^n$ and if $K$ is contained in $I$, we say that $K$ is a compact subset of $I$.

[A-28] By a conventional function we mean a function whose domain is contained in $\mathbb{R}^n$ or $\mathbb{C}^n$ and whose range is in either $\mathbb{R}^r$ or $\mathbb{C}^r$.

[A-29] Let $I$ be an open set in $\mathbb{R}^n$. By a locally integrable function on $I$ we mean a conventional function that is Lebesgue integrable on every open set $J$ in $\mathbb{R}^n$ whose closure $\bar{J}$ is a compact subset of $I$.

[A-30] A conventional function is said to be smooth if all its derivatives of all orders are continuous at all points of its domain.

[A-31] The support of a continuous function $f(t)$ defined on some open set $\mathcal{R}$ in $\mathbb{R}^n$ is the closure with respect to of the set of points $t$ where $f(t) \neq 0$ and is denoted by $\text{supp } f$.

[A-32] Let $I$ be an open subset of $\mathbb{R}^n$ or $\mathbb{C}^n$. The set $V(I)$ is said to be a Testing function space if the following conditions are satisfied.

(i) $V(I)$ consists entirely of smooth functions defined on $I$.

(ii) $V(I)$ is either complete countably multinormed space or complete countable union space.
(iii) If \( \phi_m \xrightarrow{m \to \infty} 0 \) converges in \( V(I) \) to zero, then for every non-negative integer \( k \in \mathbb{N} \), \( \frac{k^m}{m!} \phi_m \xrightarrow{m \to \infty} 0 \) converges to the zero function uniformly on every compact subset of \( I \).

\[ \text{A-33} \] A generalized function on \( I \) is any continuous linear functional on any testing function space on \( I \). In other words, \( f \) is called a generalized function if it is a member of the dual \( V'(I) \) of some testing function space \( V(I) \).

\[ \text{A-34} \] \( f \in V'(I) \) is concentrated on a subset \( \omega \) of \( I \) whenever \( \langle f, \phi \rangle = 0 \) for every \( \phi \in V(I) \) that vanishes on a neighbourhood of \( \omega \). The delta functional \( \delta(t) \) is concentrated on every set that contains the origin.

\[ \text{A-35} \] Let \( f \) be a conventional function on \( I \) such that, for every \( \phi \in V(I) \)

\[
(35.1) \quad \int_I f(t) \phi(t) \, dt
\]

converges in the lebesgue sense and, for every sequence \( \phi_m \) which converges in \( V(I) \) to zero

\[
\int_I f(t) \phi_m(t) \, dt \to 0 \quad \text{as} \quad m \to \infty
\]
Then, by setting $\langle f, \phi \rangle$ equal to (35.1), we define a member

$f$ of $V'(I)$ which is referred as a regular generalized

function in $V'(I)$ or as a function in $V'(I)$. If $f \in V'(I)$ is
called singular, if it is not regular.

[A.36] The delta functional $\delta(s-a)$ concentrated on $a \in I$

is defined by

$$\langle \delta(s-a), \phi(s) \rangle = \phi(a)$$

[A-37] A function $f(x)$ is said to be of exponential

order as $x \to \infty$, if $\lim_{x \to \infty} e^{-ax} f(x) = \text{finite quantity}$, that is,

if given a positive integer $n_0$ there exists a real number $M > 0$ such that

$$|e^{-ax} f(x)| < M, \quad \forall \quad x \geq n_0$$

Or

$$|f(x)| < M e^{ax}, \quad \forall \quad x \geq n_0$$

some times we write

$$f(x) = O(e^{ax}), \quad x \to \infty$$

[A-38] The spaces $D_k(I)$ and $D(I)$ and their duals.

Let $K$ be any compact subset of $I \subset \mathbb{R}^n$. $D_k(I)$ is a set of

all complex valued smooth functions defined on $I$ which

vanish outside of $K$. $D_k(I)$ is a linear space under the
usual definitions of addition and multiplication by scalars. The zero element of \( D_k(I) \) is the identically zero function on \( I \). The topology of \( D_k(I) \) is generated by the multinorm

\[
\lVert \psi \rVert_k = \sup_{t \in I} t^k \lvert \psi(t) \rvert, \quad \psi \in D_k(I)
\]

The space \( D_k(I) \) is a testing function space on \( I \). If \( \{k_m\}_{m=1}^{\infty} \) is a sequence of compact subset of \( I \) with the following properties

(i) \( k_m \supseteq k_{m+1}, \quad m=1,2,3,... \) (ii) Each compact subset of \( I \) is contained in one of the \( k_m \), then \( D_{k_m} (I) \supseteq D_{k_{m+1}} (I) \) and the topology on \( D_{k_m} (I) \) is stronger than the topology induced on it by \( D_{k_{m+1}} (I) \). Therefore, the countable union space \( D(I) \) is the space \( D(I) = \bigcup_{m=1}^{\infty} D_{k_m} (I) \).

The continuous linear functional on the space \( D(I) \) is called distribution on \( I \). Thus, the members of \( D'(I) \), the dual space of \( D(I) \) are the distributions.
Hahn-Banach Theorem: Let $H$ be a linear subspace of normed space $X$ and let $f$ be a continuous linear functional on $H$. Then $f$ can be extended into a continuous linear functional $F$ on $X$ and

$$\sup_{x \in H, \|x\| = 1} |f(x)| = \sup_{x \in X, \|x\| = 1} |F(x)|.$$  

The Space of $L_a^p(I)$: Let $I$ denote any open interval $a < x < b$ on the real line. A function $f(x)$ is said to be quadratically integrable on $I$ if it is a locally integrable function on $I$ such that

$$(1) \quad a^p(f) = \left( \int_a^b |f(x)|^p \, dx \right)^{1/p} < \infty.$$  

The set of all quadratically integrable functions can be partitioned into equivalence classes of such functions. Moreover, the functional $\alpha_0$ is also defined on $L_2(I)$; that is, the number that $\alpha_0$ assigns to any equivalence class is defined as the number that $\alpha_0$ assigns to any one of its members, this number being the same for all members of a given class. $L_2(I)$ is a linear space whose zero element is the class of all functions that are equal to zero almost every where on $I$. Moreover $\alpha_0$ is norm on $L_2(I)$ and is therefore a special case of a countable multinorm. $L_2(I)$ is assigned the topology generated by $\alpha_0$. It turn out that $\alpha_2(I)$ is a complete space.
REFERENCES


On Marchi–Zgrablich Transformation of
Generalized Functions

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The purpose of this paper is to extend the classical Marchi–Zgrablich
transformation of generalized functions by using the orthonormal series expansion
of generalized functions. The inversion and uniqueness theorems have been proved.
The Marchi–Zgrablich transformation of generalized functions has been applied to
find the generalized solution of the problem of vibrations in hollow circular
membranes with elastic support with initial conditions as a generalized function, by
using the operational calculus developed in this paper.

1. Introduction

Certain orthonormal series expansions of various generalized functions lead to
the so-called finite integral transformations. Zemanian (1968a,b) has extended finite
Laplace, Hermite, Jacobi and finite Hankel transformations of generalized functions
by using orthonormal series expansions of generalized functions. Dube (1976) has
extended finite Hankel transformations of generalized functions by a method which
is entirely different from that above.

In this paper we define the type of generalized functions to which the Marchi–
Zgrablich transformation has been applied. Also we shall solve the Cauchy problem
of vibrations in hollow circular membranes with initial conditions as distributions
by the application of the Marchi–Zgrablich transform of generalized functions.

We restrict ourselves to real-valued functions and real functionals. All other
results (Zemanian, 1968a,b) can be extended to complex-valued testing functions and
complex functionals. In this latter case a slightly different form for inner products
and for the numbers that functions assign to testing functions is customary.

The Marchi–Zgrablich transformation of a function $f(x)$ defined on the interval
$(a, b)$ is defined in Marchi & Zgrablich (1964) as

$$
\mathcal{M}(n) = \int_a^b x f(x) S_n(k_1, k_2, \mu_n x) \, dx
$$

whose inversion is given by

$$
f(x) = \sum_n \frac{1}{c_n} \mathcal{M}(n) S_n(k_1, k_2, \mu_n x)
$$
where
\[ S(\mu, x) = J(\mu)(Y(\mu, a) + Y(\mu, b)) - Y(\mu)(J(\mu, a) + J(\mu, b)) \]

\( J(\mu, x) \) and \( Y(\mu, x) \) are Bessel functions of the first and second kind respectively of order \( \nu \).

2. Notation and Terminology

In this work \( x \) will be a real one-dimensional variable restricted to some open interval \( I = (a, b) \) and \( n \) will be a non-negative integer. The conventional or generalized derivative of a function \( \theta \) is denoted by \( D^\theta \). The \( n \)th derivative of \( \theta \) is denoted by \( D^n \theta \). We shall say that a function \( f \) is piecewise continuous on an open interval \( I \), if for every closed bounded subinterval \( c \leq x \leq d \) of \( I \), the following conditions hold:

(i) \( f \) is continuous on \( c < x < d \) except possibly at a finite number of points \( x_n, i = 1, 2, \ldots, k \).
(ii) At these exceptional points \( x_n \), the left-hand limit \( f(x_n - \epsilon) \) and right-hand limit \( f(x_n + \epsilon) \) exist. Under this definition the function \( f(x) \) may have an infinite number of finite discontinuities on the entire interval \( I \) and may tend to infinity at either end point of \( I \).

3. The Testing Function Space \( \beta \) and Its Dual \( \beta' \)

Consider the functions \( \Psi_n(x) \) defined on \( I \) as
\[ \Psi_n(x) = \sqrt{2\pi} J(\mu_n, x) Y(\mu_n, a) - Y(\mu_n, b) \]

where \( J_n \) and \( Y_n \) are the \( n \)th-order Bessel functions of the first and second kind respectively and \( \mu_n \) denotes all positive roots of the equation
\[ J_n(\mu_n)Y_n(\mu_n) - Y_n(\mu_n)J_n(\mu_n) = 0 \]

with \( 0 < \mu_1 < \mu_2 < \ldots \) and
\[ C_n = \mu_n^{-2} \left( 1 - \frac{J(k_1, k_2, \mu_n b)^2}{J(k_1, k_2, \mu_n a)^2} \right) \]

Also let \( \eta \) denote the differential operator
\[ \eta = x^{-\nu-\frac{1}{2}} Dx^{2\nu+1} Dx^{-\nu-\frac{1}{2}}, \quad \nu \geq -\frac{1}{2} \]

The functions \( \Psi_n \) happen to be eigenfunctions of \( \eta \) that is \( \eta \Psi_n = \lambda_n \Psi_n \) where \( \lambda_n = -\mu_n^2 \). The \( \Psi_n \) comprise an orthonormal set, that is
\[ \langle \Psi_n, \Psi_m \rangle = \int_a^b \Psi_n(x) \Psi_m(x) \, dx = \begin{cases} 0 & \text{if } n \neq m \\ 1 & \text{if } n = m \end{cases} \] (3.1)
Moreover for any real-valued function \( f(x) \) having a continuous derivative on \( I \) and satisfying
\[
\int_a^b |f(x)|^2 \, dx < \infty,
\]
we have
\[
f = \sum_{n=0}^{\infty} \langle f, \Psi_n \rangle \Psi_n
\]
(3.3)
where the series is understood to converge pointwise on \( I \) As is usual when dealing with an ordinary function, \( f \) the notation \( \langle f, \Psi_n \rangle \) denotes the inner product defined by
\[
\langle f, \Psi_n \rangle = \int_a^b f(x) \Psi_n(x) \, dx
\]

We use these classical facts to construct a testing function space \( \beta \), whose dual consists of generalized functions, which can be expanded in a generalized sense into a series like (3.3)

In particular \( \beta \) consists of all smooth real-valued functions on \( I \) such that for each non-negative integer \( k \),
\[
y_k(\theta) = y_0(\eta^k \theta) = \left\{ \int_a^b [\eta^k \theta(x)]^2 \, dx \right\}^{\frac{1}{2}} < \infty
\]
(3.4)
and for each \( n \) and \( k \)
\[
\langle \eta^k \theta, \Psi_n \rangle = \langle \theta, \eta^k \Psi_n \rangle
\]
(3.5)
Both sides of above equality exist because by (3.1) and the Schwarz inequality
\[
\int_a^b |\Psi_n \eta^k \theta| \, dx \leq \left[ \int_a^b \Psi_n^2 \, dx \right]^{\frac{1}{2}} \left[ \int_a^b (\eta^k \theta)^2 \, dx \right]^{\frac{1}{2}} = y_k(\theta(\eta)) < \infty
\]
and
\[
\int_a^b |\partial^k \Psi_n| \, dx = \lambda_n^k \int_a^b |\partial^k \Psi_n| \, dx \leq \lambda_n^k \, y_0(\theta) < \infty
\]
for all \( \theta \in \beta \)

\( \beta \) is a linear space Moreover \( \{y_k\}_{n=0}^{\infty} \) is a multilinear on \( \beta \) Hence each \( y_k \) is a seminorm and in addition \( y_0 \) is clearly a norm on \( \beta \) We equip \( \beta \) with the topology that is generated by \( \{y_k\}_{n=0}^{\infty} \) and this makes \( \beta \) a countably multilinear space

Also the space \( \beta \) is Frechet space (see Theorem 1 in Zemanian, 1966) Under this formulation \( \beta \) turns out to be a testing function space

**Lemma 3.1** Every \( \Psi_n(x) \) is a member of \( \beta \)

**Proof** Since \( \eta^k \Psi_n = \lambda_n^k \Psi_n \) by (3.4) we have
\[
|y_k(\Psi_n)|^2 = \int_a^b (\eta^k \Psi_n)^2 \, dx = \lambda_n^k \int_a^b \Psi_n^2 \, dx = \lambda_n^k < \infty
\]
Also for \( n \neq m \) we have
\[
\langle \eta^k \psi_n, \psi_m \rangle = \lambda_n^k \langle \psi_n, \psi_m \rangle = 0 = \lambda_n^k \langle \psi_n, \psi_n \rangle = \langle \psi_n, \eta^k \psi_n \rangle
\]
when as for \( n = m \)
\[
\langle \eta^k \psi_n, \psi_n \rangle = \lambda_n^k \langle \psi_n, \psi_n \rangle = \langle \psi_n, \lambda_n^k \psi_n \rangle = \langle \psi_n, \eta^k \psi_n \rangle
\]
since \( \lambda_n \) is real. Hence \( \psi_n \in \beta \) for all \( n \).

The set of all continuous linear functionals on \( \beta \) is called the dual of \( \beta \) and is denoted by \( \beta' \). The members of \( \beta' \) are called generalized functions on \( \beta \). The members of \( \beta' \) not only depend on the choice of \( I \), but also on \( \eta \) and \( \psi_n \), as in \( \beta \).

**Example 3.1** Let \( f(x) \) be a locally integrable and real-valued continuous function on \( I \), whose support is compact with respect to \( I \) such that
\[
\int_a^b |f(x)|^2 \, dx < \infty
\]
Then \( f \) generates a member of \( \beta' \), through the definition
\[
\langle f, \theta \rangle = \int_a^b f(x) \theta(x) \, dx, \quad \theta \in \beta'
\]
Indeed, it is clear that (3.6) defines a functional \( f \) on \( \beta \). Its continuity can be established by the Schwarz inequality and such members of \( \beta' \) are called regular generalized functions in \( \beta' \). All other generalized functions in \( \beta' \) are called singular generalized functions.

4. The Generalized Function Space \( \beta' \)

Since the testing function space \( \beta \) is complete, so also is \( \beta' \) according to Zemanian (1968a, theorem 18.3).

We define a generalized differential operator \( \eta' \) on \( \beta' \) through the relation
\[
\langle f, \eta' \theta \rangle = \langle f, \eta \theta \rangle = \langle \eta', \theta \rangle = \langle f, \eta' \theta \rangle
\]
According to the convention stated in Zemanian (1968a, 25) \( \eta' \) is denoted by the differential expression obtained by reversing the order in which the differentiations and multiplication by \( \phi \) occur in \( \eta \), replacing each \( D \) by \(-D\) and then taking the complex conjugate of the result. But this is precisely the expression for \( \eta \) according to Zemanian (1968a, Sec 9.2, eq 4) Thus \( \eta = \eta' \) is defined as a generalized differential operator on \( \beta' \) through the equation \( \langle \eta f, \theta \rangle = \langle f, \eta' \theta \rangle \), \( f \in \beta' \), \( \theta \in \beta \). Since \( \eta \) is a continuous linear mapping of \( \beta \) into \( \beta \), it is also a continuous linear mapping of \( \beta' \) into \( \beta' \).

4.1 Some Other Properties of \( \beta' \)

(1) \( D(I) \) is obviously a subspace of \( \beta \) and convergence in \( D(I) \) implies convergence in \( \beta \). Consequently the restriction of any \( f \in \beta' \) to \( D(I) \) is a member of \( D'(I) \). Moreover convergence in \( \beta' \) implies convergence in \( D'(I) \).
(ii) Zemanian (1968a, theorem 1.8.1) asserts the following

For each \( f \in \beta' \) there exists a non-negative integer \( r \) and positive constant \( C \) such that

\[
|\langle f, \theta \rangle| \leq C \max_{0 \leq k \leq r} y_k(\theta)
\]

for every \( \theta \in \beta \). Here \( r \) and \( C \) depend on \( f \) but not on \( \theta \).

(iii) Since \( D(I) \subset \beta \subset E(I) \) and since \( D(I) \) is dense in \( E(I) \), \( \beta \) is also dense in \( E(I) \). It follows from Zemanian (1968a, corollary 1.8.2a and lemma 9.3.4) that \( E(I) \) is a subspace of \( \beta' \).

(iv) Since \( \eta \) is continuous linear mapping from \( \beta' \) into \( \beta' \), it follows that \( \eta f \in \beta' \) whenever \( f \) is a regular generalized function in \( \beta' \). It turns out that the converse of this result is also true (see Zemanian, 1966, theorem 6).

More specifically we can state a necessary and sufficient condition for \( f \) to be in \( \beta' \) is that there exists a non-negative integer \( q \) and a regular generalized function \( g \) in \( \beta' \) such that

\[
f = \eta^q g + C \quad \text{when } C \text{ is a constant.}
\]

5. Orthonormal Series Expansions and Generalized Integral Transformation

The fundamental theorem of this paper now follows. It states that any generalized function in \( \beta' \) possesses an orthonormal series expansion with respect to the \( \Psi_n \) used in the construction of \( \beta \).

**Theorem 5.1** If \( f \in \beta' \) then

\[
f = \sum_{n=0}^{\infty} \langle f, \Psi_n \rangle \Psi_n
\]

where the series converges in \( \beta' \).

**Proof** To prove the theorem we need first to prove that if \( \theta \in \beta \) then

\[
\theta = \sum_{n=0}^{\infty} \langle \theta, \Psi_n \rangle \Psi_n
\]

where the series converges in \( \beta \). For, let \( \theta \in \beta \). By (3.4) \( \eta^k \theta \) is in \( L_2(I) \) for each non-negative integer \( k \). Hence we may expand \( \eta^k \theta \) into a series of the orthonormal functions \( \Psi_n \). Using (3.5) and the fact that \( \eta \Psi_n = \lambda_n \Psi_n \) where \( \lambda_n \) is a real number we obtain

\[
\eta^k \theta = \sum_{n=0}^{\infty} \langle \eta^k \theta, \Psi_n \rangle \Psi_n = \sum_{n=0}^{\infty} \langle \theta, \eta^k \Psi_n \rangle \Psi_n = \sum_{n=0}^{\infty} \langle \theta, \lambda_n \Psi_n \rangle \Psi_n
\]

\[
= \sum_{n=0}^{\infty} \langle \theta, \Psi_n \rangle \lambda_n \Psi_n = \sum_{n=0}^{\infty} \langle \theta, \Psi_n \rangle \eta^k \Psi_n
\]

(5.2)

This series converges in \( L_2(I) \), consequently for each \( k \),

\[
y_0 \left[ \theta - \sum_{n=0}^{\infty} \langle \theta, \Psi_n \rangle \Psi_n \right] = y_0 \left[ \eta^k \theta - \sum_{n=0}^{\infty} \langle \theta, \Psi_n \rangle \eta^k \Psi_n \right] \to 0 \quad \text{as } n \to \infty
\]
Now for any \( \theta \in \beta \)
\[
(f, \theta) = \left( f, \sum_{n=0}^{\infty} (\theta, \Psi_n)\Psi_n \right) = \sum_{n=0}^{\infty} (f, \Psi_n)(\theta, \Psi_n) = \sum_{n=0}^{\infty} (f, \Psi_n)(\Psi_n, \theta) \tag{5.3}
\]

The right-hand side of (5.3) truly converges for any \( \theta \in \beta \), which means that the series in (5.1) converges in \( \beta' \) Q.E.D.

The members of \( \beta' \) lead to the generalized Marchi–Zgrablich transformation \( MZ \) defined by
\[
MZf = F(n) = \langle f, \Psi_n \rangle = \langle f, \Psi_n \rangle, \quad f \in \beta', \quad n = 0, 1, 2,
\]

Thus a linear and continuous mapping \( MZ \) maps \( f \in \beta' \) into a function \( F(n) \) defined on the set of non-negative integers.

The inverse (generalized) Marchi–Zgrablich transformation \( MZ^{-1} \) is defined by the series (5.1) which we rewrite as,
\[
MZ^{-1}F(n) = \sum_{n=0}^{\infty} F(n)\Psi_n = \sum_{n=0}^{\infty} \langle f, \Psi_n \rangle \Psi_n = f
\]

**Theorem 5.2 (Uniqueness Theorem)** If \( f, g \in \beta' \) and if their transform satisfies \( F(n) = G(n) \) for every \( n \) then \( f = g \) in the sense of equality in \( \beta' \).  

**Proof** For any \( \theta \in \beta \)
\[
\langle f, \theta \rangle - \langle g, \theta \rangle = \sum_{n=0}^{\infty} \langle f - g, \Psi_n \rangle \langle \Psi_n, \theta \rangle = \sum_{n=0}^{\infty} |F(n) - G(n)|\langle \Psi_n, \theta \rangle = 0, \quad \text{for all } n
\]

6. An Operational Calculus

Since the differential operator \( \eta \) is a continuous linear mapping of \( \beta' \) into \( \beta' \), we may write for every \( f \in \beta' \)
\[
\eta^k f = \sum_{n=0}^{\infty} (f, \Psi_n)\eta^k \Psi_n = \sum_{n=0}^{\infty} (f, \Psi_n)\lambda_n^k \Psi_n
\]

We can use this fact to solve differential equations of the form
\[
P(\eta)u = g \tag{6.1}
\]

where \( P \) is a polynomial and the given \( g \) and unknown \( u \) are required to be in \( \beta' \).  We may apply it to the Marchi–Zgrablich series expansion of \( u \) term by term to get
\[
P(\eta)u = \sum_{n=0}^{\infty} \langle u, \Psi_n \rangle P(\eta)\Psi_n = \sum_{n=0}^{\infty} \langle u, \Psi_n \rangle P(\lambda_n)\Psi_n
\]

where as always \( \lambda_n = -\mu_n^2 \)
Now applying the Marchi–Zgrablich transform $MZ$ to the equation (6.1) we obtain the equations
\[
\langle u, \Psi_n \rangle P(\lambda_n) = \langle \theta, \Psi_n \rangle, \quad \text{for } n = 0, 1, 2, \tag{6.2}
\]
If $P(\lambda_n) \neq 0$ for every $n$, we can divide (6.2) by $P(\lambda_n)$ and apply $MZ^{-1}$ to get
\[
MZ^{-1} \langle u, \Psi_n \rangle = \sum_{n=0}^{\infty} \frac{\langle \theta, \Psi_n \rangle}{P(\lambda_n)} \Psi_n
\]
\[
u = \sum_{n=0}^{\infty} \frac{\langle \theta, \Psi_n \rangle}{P(\lambda_n)} \Psi_n \tag{6.3}
\]
By Zemanian (1968a, theorem 9.6.1) and theorem 5.2, this solution $u$ exists and is unique in $\beta'$.

On the other hand if $P(\lambda_n) = 0$ for some $\lambda_n$ say $\lambda_{nk} (k = 1, 2, \ldots, m)$, then the solution (6.3) exists only if $\langle \theta, \Psi_n \rangle = 0$ for $k = 1, \ldots, m$. Moreover in this case the solution is no longer in $\beta'$ since we can add to it any complementary solution
\[
u_c = \sum_{k=1}^{m} a_k \Psi_{nk}
\]
where $a_k$ are arbitrary numbers.

7. An Application to Vibration in Hollow Circular Membranes

In this section we shall solve the Cauchy problem of vibrations in a hollow circular membrane, with initial conditions as distributions, by the application of Marchi–Zgrablich transform of generalized functions.

We shall now find the generalized solution $z(x)$ in the space $\beta'$ on the domain $\{(x, \theta) \mid a \leq x \leq b, 0 \leq \theta \leq 2\pi\}$ which satisfies the differential equation
\[
\frac{\partial^2 z(x)}{\partial x^2} + \frac{1}{x} \frac{\partial z(x)}{\partial x} + \frac{P(x, t)}{T} \frac{1}{C^2} \frac{\partial^2 z(x)}{\partial t^2} \tag{7.1}
\]
for vibrations in a membrane. Let the transversal displacement $z(x)$ be a generalized function in polar co-ordinates that does not depend on the variable $x$ but does depend parametrically upon the time variable $t$. $P(x, t)$ is the exterior pressure applied normally to the membrane, $T$ is the tension to which the membrane is submitted, and $C^2 = T/\sigma$, $\sigma$ being the mass per unit area.

We consider the boundaries $x = a$ and $x = b$ of the membrane to be supported by elastic supports submerged in a non-viscous medium. Then we have the following boundary conditions,
\[
\begin{align*}
z(a) + k_1 \frac{\partial}{\partial x} z(x) &= 0 \quad \text{for } x = a \quad \text{and all } t \tag{7.2} \\
z(b) + k_2 \frac{\partial}{\partial x} z(x) &= 0 \quad \text{for } x = b \quad \text{and all } t,
\end{align*}
\]
where \( k = T/\zeta_1 \) and \( k_2 = T/\zeta_2 \), \( \zeta_1 \) and \( \zeta_2 \) being the elastic constants at the supports given by Hook's law.

We impose the initial conditions

(a) \( z_0(x) \) converges in some generalized sense to \( g(x) \) in \( \beta' \) as \( t \to 0 \)
(b) \( \partial/\partial t z_0(x) \) converges in some generalized sense to \( h(x) \) in \( \beta' \) as \( t \to 0 \)

Taking the Marchi-Zgrablich transform of both sides of (7.1) with \( \nu = 0 \) we get

\[
-\mu_a^2 Z_n(t) + \frac{P(n, t)}{T} = \frac{1}{C^2} \frac{\partial^2 Z_n(t)}{\partial t^2}
\]

\[
\frac{\partial^2 Z_n(t)}{\partial t^2} + \mu_a^2 c^2 Z_n(t) = \frac{P(n, t)}{T}
\]  

(7.3)

where \( Z_n(t) = F(n) = \langle z_0(x), \Psi_n(x) \rangle \). After solving the equation (7.3) we get

\[
Z_n(t) = A(n) \cos \mu_a c t + B(n) \sin \mu_a c t + \frac{1}{T \mu_a c} \sin \mu_a c t \ast P(n, t)
\]  

(7.4)

where \( A(n) \) and \( B(n) \) are unknown generalized functions which do not depend on \( t \) and \( \sin \mu_a c t \ast P(n, t) \) is the convolution of functions \( \sin \mu_a c t \) and \( P(n, t) \).

In view of initial conditions (a) and (b) we have the additional conditions

\[ MZz_0(x) = MZg(x) = G(n) = \langle g(x), \Psi_n(x) \rangle, \quad \text{as} \quad t \to 0 \]

and

\[ MZ \frac{\partial}{\partial t} z_0(x) = MZh(x) = H(n) = \langle h(x), \Psi_n(x) \rangle, \quad \text{as} \quad t \to 0 \]

Hence with respect to the above initial conditions we get,

\[ A(n) = G(n) \quad \text{and} \quad B(n) = \frac{H(n)}{\mu_a c} \]

Then the equation (7.4) becomes,

\[ Z_n(t) = G(n) \cos \mu_a c t + \frac{H(n)}{\mu_a c} \sin \mu_a c t + \frac{1}{T \mu_a c} \sin \mu_a c t \ast P(n, t) \]

Finally taking the inverse transform of the above equation we obtain the solution

\[ z_0(x) = \sum_{n=0}^{\infty} \left\{ G(n) \cos \mu_a c t + \frac{H(n)}{\mu_a c} \sin \mu_a c t + \frac{1}{T \mu_a c} \sin \mu_a c t \ast P(n, t) \right\} \Psi_n(x) \]  

(7.5)

We now verify that (7.5) is truly a solution of the differential equation (7.1).

To justify this we first show that for each \( t \), \( z_0(x) \) is a member of \( \beta' \) on \( a < x < b \). Since \( g(x) \) and \( h(x) \) are members of \( \beta' \) and, as \( n \to \infty \), \( G(n) = O(n^p) \) for some integer \( P \), hence it follows that \( G(n) \) and \( H(n) \) are also members of \( \beta' \). Also it is a fact that \( \mu_a \sim n \pi / (b - a) \) as \( n \to \infty \) (McLachlan, 1955, p 15). Again for any fixed \( t \), \( \cos \mu_a c t \) and \( (\mu_a c)^{-1} \sin \mu_a c t \) are bounded and with these the series (7.5) becomes uniformly convergent on the domain \( a < x < b \) and \( 0 \leq t \leq 2 \pi \).

It is also clear that the series, obtained by applying the operators \( \eta \) for \( \nu = 0 \) and
$D_\nu^2$ separately under the summation sign of (7 5), converges uniformly in the above stated domain Thus it is valid to apply $c^2 \eta$ for $\nu = 0$ and $D_\nu^2$ separately term by term to (7 5) and, using the fact that $\eta \Psi_\nu = \lambda_\nu \Psi_\nu = -\mu_\nu^2 \Psi_\nu$, we see that (7 5) satisfies the differential equation (7 1) We shall now verify that our solution $z_\nu(x)$ satisfies initial condition (a), that is we shall show that for $\theta(x) \in \beta$,

$$\langle z_\nu(x), \theta(x) \rangle \to \langle g(x), \theta(x) \rangle \text{ as } t \to 0$$  \hspace{1cm} (7 6)

Indeed, for any fixed $t > 0$ (7 5) converges in $\beta$ according to Zemanian (1968a, Lemma 9 3 3), consequently we can form the inner product of $z_\nu(x)$ and $\theta(x)$ term by term to get,

$$\langle z_\nu(x), \theta(x) \rangle = \sum_{n=0}^{\infty} \left[ G(n) \cos \mu_\nu x + \frac{H(n)}{\mu_\nu c} \sin \mu_\nu x + \frac{1}{T \mu_\nu c} \sin \mu_\nu x \ast \tilde{p}(n, t) \right] \Psi_\nu(x), \theta(x) \rangle \hspace{1cm} (7 7)

But the functions $G(n)$ and $H(n)$ are of slow growth, while $\langle \Psi_\nu(x), \Phi(x) \rangle$ is of rapid descent, by Zemanian (1968a, Lemma 9 3 2 and 9 3 3) So (7 7) converges uniformly on $0 \leq t < \infty$ and we may pass to the limit under the summation sign as $t \to 0$ We get

$$\langle z_\nu(x), \theta(x) \rangle \to \sum_{n=0}^{\infty} G(n) \langle \Psi_\nu(x), \theta(x) \rangle = \langle g(x), \theta(x) \rangle$$

This completes our argument showing that $z_\nu(x)$ satisfies initial condition (a)

Now we verify that $z_\nu(x)$ satisfies also the initial condition (b), that is we shall show that for any $\theta \in \beta$

$$\langle \frac{\partial}{\partial t} z_\nu(x), \theta(x) \rangle \to \langle h(x), \theta(x) \rangle \text{ as } t \to 0$$

For $\theta(x) \in \beta$ we get

$$\left\langle \frac{\partial}{\partial t} z_\nu(x), \theta(x) \right\rangle = \sum_{n=0}^{\infty} \left[ G(n) \cos \mu_\nu x + \frac{H(n)}{\mu_\nu c} \sin \mu_\nu x + \frac{1}{T \mu_\nu c} \sin \mu_\nu x \ast \tilde{p}(n, t) \right] \Psi_\nu(x), \theta(x) \rangle$$
\[
\left\langle \frac{\partial}{\partial t} z_t(x), \theta(x) \right\rangle \rightarrow \sum_{n=0}^{\infty} H(n) \left\langle \Psi_n(x), \theta(x) \right\rangle \text{ as } t \to 0 \\
= \left\langle h(x), \theta(x) \right\rangle
\]

This shows that $z_t(x)$ satisfies the initial condition (b).

This completes our verification of (7.5) as a solution of the differential equation (7.1). This solution is unique in the sense of equality over $\beta^*$ in view of uniqueness theorem 5.2.

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