CHAPTER VI
6.1) Introduction: In the present chapter we have discussed the representation of Marchi-Zgrablich transformable generalized functions. This is one of the most interesting and important problems in the theory of generalized functions, which are expressing them in terms of differential operator acting on functions or on measures. Koh E.L. 1970 [1], Pandey J.N. [2] and Sonavane K.R. [3] etc., have investigated the representation of different kinds of generalized functions.

6.2) The Representation Theorem:

A representation theorem for Marchi-Zgrablich transformable generalized function will now be proved. The method of proof is analogous to the method employed in a structure theorem for Schwartz distributions [4 pp. 253-260].

Theorem 6.2.1: Let $f$ be an arbitrary element of $M^{'}_{m,1}(I)$. Then $f$ can be represented as a finite sum,

\begin{equation}
(6.2.1) \quad f = \sum_{i=0}^{k} \mathcal{G}_{i} \left( \frac{d^{i}}{dx^{i}} \right) \left[ \xi(x)p_{i}(x)F_{i}(x) \right]
\end{equation}

where $F_{i}(x)$ are continuous functions on $I$, $p_{i}(x)$ are
polynomial of degree $k$, and $g_i$ are the Lebesgue measurable functions defined on $I$.

Proof: For every $f \in M_m(I)$ there exists a non-negative integer $r$ and positive constant $M$ such that for all $f(x) \in D(I)$, $M_m$, $\cup(I)$,

$$(6.2.2) \quad |\langle f, \phi \rangle| \leq M \max_{0 \leq k < r} \sup_{x \in I} \left| \sum_{j=0}^{k} x^{-j} \phi(x) \right|.$$

$D(I)$ being the space of all smooth functions with compact support on $I$. Expanding

$$(6.2.3) \quad |\langle f, \phi \rangle| \leq M \max_{0 \leq k < r} \sup_{x \in I} \left| \sum_{i=0}^{2k} a_{2k,i} x^{-2k+i-1} \frac{d^i}{dx^i}(\phi(x)) \right|,$$

where $a_{2k,i}$ are constants depending on the value of $\cup$,

and $a_{2k,2k} = 1$. Hence we have

$$(6.2.4) \quad |\langle f, \phi \rangle| \leq M \max_{0 \leq k < r} \sup_{x \in I} \left| \sum_{i=0}^{2k} a_{2k,i} x^{-2k+i-1} \frac{d^i}{dx^i}(\phi(x)) \right|,$$

where $\Gamma' = \max_{0 \leq i < 2k} \max_{0 \leq k < r} |a_{2k,i}|$. 


Not set

\( (6.2.5) \quad \phi_r(x) = \zeta(x) x^{-2r+1-1} \phi(x), \quad i \leq 2r. \)

\( \phi(x) \in \mathcal{D}(I), \) since \( \phi(x) \in \mathcal{D}(I) \). In fact \( \phi(x) \to \phi_r(x) \) is one to one linear mapping of \( \mathcal{D}(I) \) onto itself. Then

\[
\phi(x) = \left[ \zeta(x) \right]^{-1} 2r-i+1 x^{-m+2r-i+1} \frac{d}{dx} \phi_r(x),
\]

and therefore,

\( (6.2.6) \quad \frac{d\phi}{dx} = \left( \zeta(x) \right)^{-1} 2r-i+1 \left[ (-m+2r-i+1) x^{-1} + \frac{d}{dx} \right] \phi_r(x). \)

Let \( \text{supp } \phi(x) = \text{supp } \phi_r(x) = [A, B] \). Then

\[
\left| \frac{d\phi}{dx} \right| \leq \left| \left( \zeta(x) \right)^{-1} 2r-i+1 \left[ a_{i,r} x^{-1} + \frac{d}{dx} \right] \phi_r(x) \right|,
\]

where \( a_{i,r} = 2r - m - i+1 \)

\[
\left| \frac{d\phi}{dx} \right| \leq A_{i,r} \left( \zeta(x) \right)^{-1} 2r-i+1 \left| \phi_r(x) \right| + \left| \frac{d\phi_r(x)}{dx} \right|
\]

where \( A_{i,r} = \max \left[ a_{i,r} x^{-1}, 1 \right] \).

Hence by induction we see that
(6.2.7) \( \left| \frac{d^i \phi(x)}{dx^i} \right| \leq A_i, f (\xi(x))^{-1} 2^{r-i+1} \sum_{s=0}^{i} \frac{d^s \phi_r(x)}{dx^s} \). 

Substituting (6.2.7) in (6.2.4) we get

(6.2.8) \( \left| <f, \phi> \right| \leq M A_{i, f} \max_{0 \leq i \leq 2r} \sup_{x \in I} \sum_{s=0}^{i} \frac{d^s \phi_r(x)}{dx^s} \) 

\( \leq M' \max_{0 \leq i \leq 2r} \sup_{x \in I} \frac{d^i \phi_r(x)}{dx^i} \).

where \( M' \) and \( M'' \) are constants.

Now we can write for every \( \psi(x) \in D(I) \),

(6.2.9) \( \sup_{x \in I} \left| \psi(x) \right| \leq \sup_{x \in I} \int_{a}^{b} \frac{d \psi(x)}{dx} dx = \left\| \frac{d \psi(x)}{dx} \right\|_{L^1(a,b)} \).

where \( L^1(a,b) \) is the space of all equivalence classes of Riemann integrable functions on \( I \).

The bound (6.2.9) enables us to write (6.2.8) as

(6.2.10) \( \left| <f, \phi> \right| \leq M'' \max_{0 \leq i \leq 2r} \sup_{x \in I} \left\| \frac{d^i \phi_r(x)}{dx^i} \right\| \left\| \psi \right\|_{L^1(a,b)} \).
We consider the linear one to one mapping,

\[ T : D(I) \to \left((a,b) \right) \]

\[ \phi(x) \to \left\{ \frac{d^i \phi(x)}{dx^i} \right\}, \quad 1 \leq i \leq 2r+1. \]

Since \( D(I) \) is linear subspace of \( (a,b) \). In view of (6.2.10) we see that the linear functional \( T; \phi \to \langle f, \phi \rangle \) is continuous on \( T(D(I)) \) for the topology induced by \( (a,b) \). Hence by Hahn–Banach theorem it can be extended as a continuous linear functional on the whole of \( (a,b) \). But the dual of \( (a,b) \) is isomorphic with \( \infty(a,b) \) [4 p. 254]. The space of all bounded measurable functions on \( I \) such that for every \( f \in \infty(a,b) \) there exist \( M \) such that \( |f| \leq M \) almost everywhere.

Therefore on account of Riesz-representation theorem

[see appendix A-45] there exist \( M \)-bounded measurable function \( g_i(x), 1 \leq i \leq 2r+1 \) such that

\[(6.2.11) \quad \langle f, \phi \rangle = \sum_{i=1}^{2r+1} g_i(x), \frac{d^i}{dx^i} \phi(x) \]

Recalling (6.2.5) we get

\[(6.2.12) \quad \langle f, \phi \rangle = \sum_{i=1}^{2r+1} \frac{d^i}{dx^i} g_i(x), \frac{d^{2r+i-1}}{dx^{2r+i-1}} \phi(x) \]

\[ = \sum_{i=1}^{2r+1} (-1)^i \frac{d^i}{dx^i} g_i(x), \frac{d^{2r+i-1}}{dx^{2r+i-1}} \phi(x) \]
Therefore

\[ f = \sum_{i=1}^{2r+1} (-1)^i \mathcal{C}_i(x) x^{\frac{-2r+i-1}{i}} \frac{d^i}{dx^i} g_1(x). \]

For each \( i \) we set

\[ h_i(x) = (-1)^i \int_0^x g_i(s) ds, \quad \text{since} \quad g_i \in C^\infty(a,b). \]

Then the functions \( h_i \) are also continuous on \( I \) and

\[ h_i(x) \leq \int_0^x |g_i(s)| ds \]

\[ \leq |x| \max |g_1(x)| \]

\[ = |x| \| g_1(x) \|_{C^\infty(a,b)} \]

Further \( g_i(x) = (-1)^i \frac{d^i}{dx^i} h_i(x) \) implies that

\[ f = \sum_{i=1}^{2r+1} \mathcal{C}_i(x) x^{\frac{-2r+i-1}{i}} \frac{d^i}{dx^i} (h_1(x)). \]

By letting \( 2r+1 = k \) and using \( k \)th differential formulae

\[ u(t) \left( \frac{\partial}{\partial t} \right)^k h_i = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \left[ u^j h_i \right]^{k-j}. \]
and

\[(ab)^k = \sum_{i=0}^{k} \binom{k}{i} a^{k-i} b^i.\]

We can write (6.2.14) as in (6.2.1) where \(F_1(x)\) are continuous functions of \(h_1(x)\) and are therefore \(p_1(x)\) are polynomials of degree \(k\).
REFERENCES

[1] Kohl, E.L.  A representation theorem of Hankel transformable generalized functions,

[2] Pandey, J.N.  A representation theorem for a class of generalized functions,
SIAM J. Math. Anal. 2 (1972),
PP. 286-289.

[3] Sonvane, K.R.  Some properties of generalized functions and generalized integral transform,

[4] Treves, F.  Topological vector space,
Distributions and kernels,