CHAPTER IV

Instability Properties and associated
Characteristics of the Disturbances with Quasigeostrophic
Model.

In this chapter we will discuss the eigen-values and
eigen-functions of the matrix system of the baroclinic quasi-
geostrophic equations enunciated in Chapter III.

4.1 Effect of static stability, coriolis parameter and basic
wind shear upon instability.

Non-trivial solutions for the perturbations of the quasigeo-
strophic matrix system (3.15) can be obtained by solving the
frequency equation (5.14).

\[ \Delta^2 (a\mu^4 + 2\mu^2) + \Delta \left[ 2(a\beta^2 + \beta) \right] + \mu^2 (2\mu^2 - a\mu^4) + \beta^2 = 0. \]

where

\[ \alpha = \frac{\beta^2 \Delta_{\mu}^2}{\mu^2} \]

\[ \Delta = c - \Omega \]

the phase speed (may be complex) relative to the basic
current of the disturbance of wavelength \( \lambda (L = \frac{2\pi}{\mu}) \) can be
obtained by solving the equation (5.14) with given basic state
parameters such as \( s_2, \ell, \beta \) and \( \Omega \). Keeping the same basic state
parameters, the phase speeds of different scales of disturbances
can be obtained by varying the wavelength \( L \). This calculation is
repeated with different magnitudes of basic zonal wind shear.
Denoting this as a single set of results, we obtained several sets of results corresponding to different values of basic state static stability and latitude. Results were obtained over a spectrum of wavelengths ranging from 1000 km. to 20,000 km. with increments of 500 km. The magnitude of the basic zonal wind shear was varied from 2 m. sec\(^{-1}\) (100 mb)\(^{-1}\). (or approximately 1 m sec\(^{-1}\) km\(^{-1}\)) to 30 m sec\(^{-1}\). (100 mb)\(^{-1}\) (or \(\gg\) 15 m sec\(^{-1}\) km\(^{-1}\)).

Basic state static stability at 500 mb was varied in steps with 0.5, 2.0, 4.0 and 10.0 of M.T.S. units. For convenience static stability in M.T.S. units and its related lapse rates are shown in Table 4.1

**Table 4.1**

<table>
<thead>
<tr>
<th>(\bar{S}_2) (m^2 \text{Sec}^2 \text{cm}^2)</th>
<th>(-\frac{\partial \theta}{\partial p} \text{°C cm}^{-1})</th>
<th>(\frac{\partial T}{\partial p} \text{°C cm}^{-1})</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.25</td>
<td>0.052</td>
<td>1.498</td>
</tr>
<tr>
<td>0.5</td>
<td>0.105</td>
<td>1.45</td>
</tr>
<tr>
<td>1.0</td>
<td>0.21</td>
<td>1.37</td>
</tr>
<tr>
<td>2.0</td>
<td>0.42</td>
<td>1.19</td>
</tr>
<tr>
<td>4.0</td>
<td>0.84</td>
<td>0.845</td>
</tr>
<tr>
<td>6.0</td>
<td>1.26</td>
<td>0.477</td>
</tr>
<tr>
<td>10.0</td>
<td>2.12</td>
<td>-0.2</td>
</tr>
</tbody>
</table>
With the combination of above mentioned basic state parameters, the frequency equation is solved separately over three latitudes 60°N, 45°N and 15°N.

In the discussion of equation (3.15) we mentioned that the quadratic equation in \( \Delta \) will give real roots with short wavelengths as well as with long wavelengths. We know that with real solutions, the amplitude of the disturbances neither grows nor decreases with time and the disturbances are then called neutral. However, when \( \Delta \) is complex its conjugate necessarily becomes also a solution, the real part of the solution being the propagation velocity of the disturbance relative to the basic current and the imaginary part being the growth rate of the disturbance. The amplification rate of the disturbance is given by \( \mu \Delta_i \), where \( \mu \) is the zonal wave number \( \frac{-2\pi}{L} \) and \( \Delta_i \) is the imaginary part of the complex solution.

The system of equations (3.11), (3.12) and (3.15) being linear, we can superpose the two solutions corresponding to the two eigenvalues \( \Delta_1 \) and \( \Delta_2 \) for the unknown perturbations of wave number \( \mu \).

\[
\psi_1' = \psi_{1,1}' e^{\mu \Delta_1 z \textbf{t}} e^{i \mu [x - (\Delta_1 k + \Omega) t]} + \psi_{1,2}' e^{\mu \Delta_2 z \textbf{t}} e^{i \mu [x - (\Delta_2 k + \Omega) t]}
\]

Similarly for

\[
\psi_2' = \psi_{2,1}' e^{\mu \Delta_1 z \textbf{t}} e^{i \mu [x - (\Delta_1 k + \Omega) t]} + \psi_{2,2}' e^{\mu \Delta_2 z \textbf{t}} e^{i \mu [x - (\Delta_2 k + \Omega) t]}
\]

\[
\omega_1' = \omega_{1,1}' e^{\mu \Delta_1 z \textbf{t}} e^{i \mu [x - (\Delta_1 k + \Omega) t]} + \omega_{1,2}' e^{\mu \Delta_2 z \textbf{t}} e^{i \mu [x - (\Delta_2 k + \Omega) t]}
\]

\[
\omega_2' = \omega_{2,1}' e^{\mu \Delta_1 z \textbf{t}} e^{i \mu [x - (\Delta_1 k + \Omega) t]} + \omega_{2,2}' e^{\mu \Delta_2 z \textbf{t}} e^{i \mu [x - (\Delta_2 k + \Omega) t]}
\]
From these expressions one can notice that in the case of the complex solutions for eigen values, \( \Delta_{2k} = \Delta_{1k} \) and \( \Delta_{2i} = -\Delta_{1i} \) both amplifying and damping modes contribute to the perturbation. The damped component of wave would vanish with time leaving the perturbation field from the amplifying mode only. Hence, we can limit our discussions to the amplifying and the neutral modes only.

In order to know the degree of instability of the disturbance at different wavelengths with reference to the given basic state parameters, the time required for the disturbance to double its amplitude, generally known as 'doubling time' is calculated by the following formula

\[
\tau_D = \frac{\log_2 e^2}{\mu \Delta_i} \quad \text{--- (4.1)}
\]

Obviously \( \tau_D \) is infinity for neutral disturbances.

The doubling time curves are presented as functions of wavelength and basic zonal wind shear, for each set of basic state parameters. These figures constitute the stability diagrams for the disturbances. Figures (4.1) to (4.5) are stability diagrams corresponding to four cases, those are (i) with \( \bar{S}_2 = 2 \) MTS units at 60\(^\circ\)N, (ii) with \( \bar{S}_z = 2 \) MTS units at 45\(^\circ\)N, (iii) with \( \bar{S}_z = 4 \) MTS units at 45\(^\circ\)N, (iv) with \( \bar{S}_z = 2 \) MTS units at 15\(^\circ\)N and (v) with \( \bar{S}_z = 0.5 \) MTS units at 15\(^\circ\)N respectively. In the diagrams the solid curves depict the doubling time expressed in days. The curve

* for convenience we mentioned in all the figures \( \bar{S}_z \) as \( \delta \) only.
FIG. 4.1 WITH $S = 2$ MTS UNITS, AT 60° LAT.
FIG. 4.3 WITH $S = 4$ MTS UNITS, AT 45° LAT.

--- CURVE OF MAXIMUM INSTABILITY

DOUBLING TIME IN DAYS FOR DISTURBANCES

WAVE LENGTH IN 1000'S KM

WIND SHEAR IN M/SEC (100 MPH)
FIG. 4.4 WITH $S = 2$ MTS UNITS, AT 15° LAT.
FIG. 4-5 WITH S = 0.5 MTS UNITS, AT 15° LATITUDE

DOUBLING TIME IN DAYS FOR DISTURBANCES

--- CURVE OF MAXIMUM INSTABILITY

WAVELENGTH IN 1000'S KM

WIND SHEAR IN m SEC (100 mb)

32 28 24 20 16 12 8 4 18 17 15 14 12 11 10 9 8 7 6 5 4 3 2

-77.
designated with infinity is a neutral curve which separates
(inclusive of itself) stable (or neutral) and unstable (or ampli-
fi ed) disturbances.

Before drawing conclusions from the above figures, one must
remember the following limitations:

(a) The linear equations of our model are based on $x, \gamma, \beta$
and $t$ co-ordinate system. The only effect of spherical nature of
the earth considered is through the variation with latitude of the
vertical component of earth's rotation. Such $\beta$-plane approxi-
mation can not be justified *rigorously* for atmospheric motions whose
scale is large compared to the radius of the earth.

(b) In drawing detailed conclusions we must limit our dis-
cussion with basic wind shear of magnitude less than 12 m sec$^{-1}$
$(100 \text{mb})^{-1}$ or ($\approx 6$ m sec$^{-1}$ km$^{-1}$).

(c) Replacement of differential quotients by difference
quotients in the vertical in our linearised equations generally
involve truncation errors. But the changes in growth rate due to the
increased number of freedom in vertical do not seem to be significant
from a comparative study of growth rates of disturbances among multi-
level quasigeostrophic models of Matsumoto (1962).

The stability diagrams show that there exists a critical basic
zonal wind shear in the vertical below which no disturbance could be
unstable. The doubling time curves show that, for a given vertical
shear, there is a particular wavelength for which doubling time
is minimum. This particular wave grows more rapidly than all the others,
and therefore may be called the most unstable disturbance. The
wavelengths of these most unstable waves for different values
of wind shear are indicated by the dashed curve. This curve shows
that the wavelength of the most unstable disturbance increases
with the increase of wind shear. However, the curve becomes
nearly vertical for large values of wind shear. For a given
wind shear let us define the 'short wave cut off' as the shortest
wavelength in the diagram below which all waves are stable and
similarly the 'Long wave cut off' as the longest wavelength above
which all waves are stable.

The general pattern of stability diagrams and magnitudes of
growth rate and wave length of the unstable disturbances are in
agreement with the results of the similar studies by Phillips
(1954), Eliassen (1952), Thompson (1961), Matsumoto (1962) and
Gates (1961). All of them analysed at 45°N with two level models.
The results from the frequency equation do not depend upon the sign
of basic zonal wind shear. In short wave part, the short wave cut
off has generally no variation with wind shear. But in long wave
part, the long wave cut off shifts towards longer wave lengths with
the increase of wind shear. The figures (4.1), (4.2) and (4.4)
correspond to different latitudes but with same static stability.
An examination of these figures show that with decrease of latitude
the doubling time for any unstable disturbance increases; the short
wave cut off and the curve of maximum instability shift towards
longer wavelengths, the long wave cut off moves to shorter wavelengths, and the critical wind shear increases. Thus the decrease of latitude has a general stabilising influence on the disturbances. An examination of figures 4.2, 4.3 show that with increase of static stability, the long wave cut off remains practically at the same wavelength but the short wave cut off shifts towards higher wavelengths. This suggests that the static stability is chiefly effective in stabilising short waves. Moreover, the increase of static stability increases the critical wind shear and generally increases the doubling time required for the unstable disturbances. The same deductions may be made from figures (4.4) and (4.5). We may thus conclude that the decrease of latitude and increase of static stability are effective in bringing stability for short waves and in increasing the critical wind shear. The stability of long waves for a fixed wind shear is controlled by decrease of latitude (or increase of $\beta$).

Aihara (1959), Wiin-Nielsen (1962), Murakami (1964) and some other authors examined quasigeostrophic baroclinic instability problem by means of initial value method. The method consists of the investigation upon the conditions under which the energy of disturbance increases or decreases with respect to time. If the kinetic energy of the disturbance tends to increase, decrease, or remain a constant with respect to time, the disturbance is respectively called unstable, stable or neutral. Aihara (1959) compared the stability diagrams of eigen value and initial value methods and found them to agree closely. Generally the shortwave cutoff, longwave cutoff and wavelength of maximum instability are the parameters that
are comparable, and they have shown close agreement with the results of eigen-value method. A comparison of the instability diagram (4.3), with the results of two level quasigeostrophic initial value problem of Murakami (1964), for a wind shear 10 m sec\(^{-1}\) (100 mb\(^{-1}\)) and a static stability of 5.46 mts units, at 40\(^\circ\)N, shows a short wave cut off at 4000 km in agreement with our result. The wavelength of maximum instability occurs at 6500 km in Fig. 4.3 and at 6000 km in Murakami's study.

The stability diagram from the initial value method consists a separation of unstable (or amplified) solutions from stable (or damped) solutions by a neutral curve. Physically our instability diagram should be considered as, a neutral curve (\(\infty\) curve) separating amplified solutions (within the area of \(\infty\) curve) and damped solutions (outside \(\infty\) curve).

From Chapter II, we know that the kinetic energy of an unstable disturbance grows with time due to the effect of release of eddy available potential energy provided by the direct thermal circulation in the sonal plane. Similarly a stable disturbance looses its kinetic energy due to the process of indirect thermal circulation.

Now from the quasigeostrophic approximation (\(\frac{f}{\rho} v' = \frac{\omega}{\omega^2} \phi'\)) \(v'\) will be positive (negative) east of the trough (ridge) of a wave in the geopotential field.
The \( \omega' \) equation can be derived from the quasigeostrophic linearised equations (3.2), (3.3) and (3.4). Differentiating (3.2) with respect to \( \phi \), applying horizontal Laplacian operator (in \( x \) only because our perturbations are \( y \) independent) to equation (3.4) and subtracting the first of these equations from the second, and making use of (3.3) we obtain

\[
\frac{\partial^2 \omega'}{\partial x^2} + \frac{R^2}{s} \frac{\partial^2 \omega'}{\partial \phi^2} = \frac{R}{s} \left\{ 2 \left( \frac{\partial u}{\partial \phi} \right) \frac{\partial \omega'}{\partial x^2} + \frac{\partial^2 \omega'}{\partial \phi^2} \right\}
\]

Finite differencing the above equation over two level model and using harmonic wave solutions for the perturbations, we obtain

\[
\omega'_x = \frac{\frac{\partial^2 \omega'}{\partial x^2}}{2 \left( \frac{\partial^2 \omega'}{\partial \phi^2} + \frac{s^2}{\mu^2} \right)} \left\{ 2 \left( \frac{\partial^2 \omega'}{\partial \phi^2} \right) \omega'_x - \frac{\partial}{\partial \phi} \left( \frac{\partial \omega'}{\partial \phi} \right) \right\} \quad (4.2)
\]

The quantities at level 2 (i.e. 500 mb level) are denoted with subscript 2.

With a sufficiently strong wind shear and short/moderate waves, the second term in the bracket is small compared with the first term. When \( \left( \frac{\partial u}{\partial \phi} \right) \) is negative, a southerly wind is associated with ascending motion and a northerly one with descending motion.

Thus in a quasigeostrophic geopotential wave, ascending (descending) motion occurs east of the trough (ridge). Referring to the thermodynamic equation (2.32)

\[
\frac{\partial}{\partial t} \left( \frac{\partial \phi}{\partial \phi} \right) = -u \frac{\partial}{\partial x} \left( \frac{\partial \phi}{\partial \phi} \right) + w \frac{\partial u}{\partial \phi} - s \omega' \quad (2.32)
\]

and its discussion in Chapter II (pages 30-32), it is easily seen
that in a quasigeostrophic model with statically stable atmosphere, 
the changes of temperature produced by vertical motion are of 
opposite kind to those produced by meridional advection. Thus 
the second term of (2.32) brings direct thermal circulation (warm 
air rising ahead of the trough and cold air sinking ahead of the 
ridge) whereas the third term brings indirect thermal circulation. 
Thus static stability which enters the third term has a stabilising 
effect on short waves. The wind shear which can make the second 
term dominate over the third term for a given \( f \) and \( S \) may be called 
critical wind shear, below which no wave could be unstable. The 
critical wind shear obviously increases with the increase of static 
stability and with the decrease of latitude. Moreover, with fixed 
static stability and wind shear, the decrease of latitude has a 
tendency to stabilise the waves.

Studies of Kuo (1965), Aihara (1964) and ours (section 4,5) 
revealed that \( \psi' \) increases with height and that the increase will 
be significant for long waves. Moreover, from the observational 
study (over North America and over Japan), \( \psi' \) was found to increase 
upto 200 mb level, Matsumoto (1956). Thus the term \( \frac{\partial \psi'}{\partial p} \) in equation 
(4.2) is generally negative. For a sufficiently long disturbance 
\( \beta \) term can dominate the first term in equation (4.2). Now ahead 
of the trough of geopotential disturbance, with \( \psi' \) positive and \( \frac{\partial \psi'}{\partial p} \) 
negative, \( \omega' \) will become positive. Thus with the domination of second 
term in (4.2) descending motion occurs ahead of the trough and the 
reverse occurs ahead of the ridge. The temperature change (from 2.32) 
produced by vertical motion is in the same sense as that produced
by meridional advection term. Thus the air descending ahead of
the trough is warm and air ascending ahead of the ridge is cold.
These disturbances will stabilise themselves by losing kinetic
energy through indirect thermal circulation. For long waves to be
stable, both the terms (2nd and 3rd) of (2.52) must be of the
same sign. So with nonzero and non negative static stability,
the criteria of stability for long waves essentially lies in
equation (4.2)*. So the long wave cutoff is a function of wind
shear and latitude only. As seen in all the instability diagrams
at a fixed latitude the long wave cutoff increases with the wind
shear. Similarly at a fixed wind shear the long wave cutoff
decreases with the increase of $\beta$ (or stabilising effect for long
waves is more pronounced at lower latitudes).

4.2 Propagation velocities for the disturbances

The equation (3.15) expresses the dependency of the propagation
velocity of the disturbance on physical parameters characterising
the basic state of the atmosphere, such as $f_s(\beta, s)$ and $u_\infty$. From
the same equation, we can say that propagation velocity (real part
of $\omega$) for the unstable disturbance is independent of wind shear.
The same phenomenon was noted for unstable disturbances from balanced

*The assumption of zero value for $\frac{\partial \psi}{\partial \phi}$ provides the answer for the
absence of long wave cutoff in the studies of initial value problem
by Murakami (1964) and Wiin-Nielsen (1962)
Unstable wave solutions from frequency equation (5.14) are given in Table 4.2 for few cases. The general order of propagation velocity is 10 m sec\(^{-1}\). For a fixed scale of disturbance the propagation velocity seems to increase with the decreasing of latitude. The general increase of velocity (with respect to basic current) with wavelength is in agreement with the long wave theory.

Disturbances of wavelength approximately of 6000 km. with the propagation velocity of order 10 m sec\(^{-1}\) and with the time required for doubling the amplitude of the order of one day (doubling time for 6000 km. is 1.1 day from Figure 4.5) is generally observed phenomenon in middle latitude region.
**TABLE 4.2**

Unstable wave solutions of the frequency equation for
lat. $45^\circ$ and $60^\circ$ velocities are given in m sec$^{-1}$

<table>
<thead>
<tr>
<th>Wave length in km</th>
<th>$\Delta_1, \Delta_2$ m sec$^{-1}$</th>
<th>$\Delta, \Delta_2$ m sec$^{-1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5000</td>
<td>$-7.29 + 4.28 i$</td>
<td>$-4.72 + 8.36 i$</td>
</tr>
<tr>
<td>5500</td>
<td>$-8.54 + 5.73 i$</td>
<td>$-5.54 + 9.20 i$</td>
</tr>
<tr>
<td>6000</td>
<td>$-9.87 + 6.59 i$</td>
<td>$-6.41 + 9.35 i$</td>
</tr>
<tr>
<td>6500</td>
<td>$-11.28 + 6.90 i$</td>
<td>$-7.35 + 10.06 i$</td>
</tr>
<tr>
<td>7000</td>
<td>$-12.77 + 6.76 i$</td>
<td>$-8.55 + 10.13 i$</td>
</tr>
<tr>
<td>7500</td>
<td>$-14.35 + 6.06 i$</td>
<td>$-9.41 + 10.04 i$</td>
</tr>
<tr>
<td>8000</td>
<td>$-16.01 + 4.58 i$</td>
<td>$-10.55 + 9.69 i$</td>
</tr>
</tbody>
</table>

With $\tilde{E}_x = 4$ MTS units and basic wind shear of 6 m sec$^{-1}$ (100 mb)$^{-1}$
4.3 Vertical structure of the disturbance

Knowing the values of $\Delta$ at different wavelengths with respect to a set of given basic state parameters, we can solve the original system of equations (3.11), (3.12) and (3.13) in unknowns $\hat{\psi}_1$, $\hat{\psi}_3$ and $\hat{\omega}_2$. The system of equations (3.11), (3.12) and (3.13) may be written as follows:

\[
\begin{align*}
(\Delta - U_\ast + \frac{\beta}{\mu_2}) i \mu_1^3 \hat{\phi}_1 - \frac{f^2}{\Delta \mu} \hat{\omega}_2 &= 0 \\
(\Delta + U_\ast + \frac{\beta}{\mu_2}) i \mu_3^3 \hat{\phi}_3 + \frac{f^2}{\Delta \mu} \hat{\omega}_2 &= 0 \\
\left( \Delta + U_\ast \right) \hat{\phi}_1 - \left( \Delta - U_\ast \right) \hat{\phi}_3 &= 0
\end{align*}
\]

(4.3)

Here all the coefficients of the unknowns $\hat{\phi}_1$, $\hat{\phi}_3$ and $\hat{\omega}_2$ are known with a corresponding value of $\Delta$ from the frequency equation (3.14). The set of equations (4.3) forms a simultaneous system of homogeneous algebraic equations. By arbitrarily fixing one of the unknowns we can express the rest of the unknowns in terms of the former one. As we are interested in knowing the structure of the unstable disturbance, we will use the eigen value $\Delta$ corresponding to the amplifying mode only.

Let us arbitrarily fix the amplitude of the geopotential perturbation at 750 mb (i.e. $\hat{\phi}_3$) as 50 m sec$^{-2}$ (i.e. perturbation contour height of 750 mb is approximately 5 meters). Now $\hat{\phi}_3$ being real the $\hat{\phi}_3'$ harmonic wave perturbation will have zero phase angle. This provides us to find out the amplitude and phases of all the perturbations with respect to $\hat{\phi}_3'$. Now the above set of equations turn
out to be three nonhomogeneous equations in two unknowns. Any
two nonhomogeneous equations of the above set will be useful for
solving and the third equation can be used as a check for the
solutions.

The equations for the unknowns from (4.3) are:

\[ \phi''_{1,2} = -\phi_3 \left\{ \frac{(\Delta_2 + \frac{3}{\mu^2} - U_x + \Delta_2^2)}{(\Delta_2 - U_x + \frac{3}{\mu^2})^2 + \Delta_2^2} \right\} \]

\[ \phi''_1 = \phi_3 \left\{ \frac{\Delta_2 U_x + \Delta_2^2}{(\Delta_2 - U_x + \frac{3}{\mu^2})^2 + \Delta_2^2} \right\} \]

\[ \phi''_2 = \phi_3 \left\{ \frac{\Delta_2 U_x}{(\Delta_2 - U_x + \frac{3}{\mu^2})^2 + \Delta_2^2} \right\} \]

\[ \phi''_3 = \frac{\mu^2 \Delta_2}{f^2} \left( \Delta_2 U_x + \frac{3}{\mu^2} \right) \]

Subscripts \( \cdot \) and \( \cdot \) denote real and imaginary parts respectively.
Thus the solutions are obtained from (4.4) and checked with the
third equation in (4.5) for the most unstable disturbances with
different basic state parameters.

Knowing the complex quantity of each unknown, we can write
exact solutions for the perturbations in accordance with (5.10) as

\[ \phi'_i = \phi_i e^{\mu \Delta_2 t} e^{i \frac{3}{\mu^2} (x - (\Delta_2 + \mu^2))} \]

\[ = (\phi''_{1,2} + i \phi''_{1,2}) e^{\mu \Delta_2 t} \left\{ \cos \mu x + i \sin \mu x \right\} \] \( \cdot \) (4.5)

where \( x = [x - (\Delta_2 + \mu^2)] \)
As usual the perturbations are to be identified with the real part of (4.5), we find therefore

\[ \phi'_{1} = e^{\Delta t} \left[ \phi'_{1} \cos \mu x - \phi'_{1, z} \sin \mu x \right] \]

Similarly for \( \omega'_{2} \) and \( \phi'_{3} \)

\[ \omega'_{2} = e^{\Delta t} \left[ \omega_{2, a} \cos \mu x - \omega_{2, z} \sin \mu x \right] \]
\[ \phi'_{3} = e^{\Delta t} \phi_{3} \cos \mu x \]

These perturbation parameters can be expressed as follows

\[ \phi'_{1} = A_{t} \phi_{1} \cos (\mu x + \delta_{1}) \]
\[ \omega'_{2} = A_{t} \omega_{2} \cos (\mu x + \delta_{2}) \]
\[ \phi'_{3} = A_{t} \phi_{3} \cos \mu x \] \hspace{1cm} (4.6)

Where \( A_{t} \phi_{1}, A_{t} \omega_{2} \) and \( A_{t} \phi_{3} \) are amplitudes at time \( t \) and \( \delta_{1} \) and \( \delta_{2} \) are initial phases of the corresponding perturbations. Here \( A_{t} \) is the rate of amplification parameter given by \( \exp (\mu \Delta t) \) for the perturbations and it is a function of wavelength and time.

Moreover \( \phi'_{1}, \omega_{2} \) and \( \phi_{3} \) are real and positive and are given as follows:

\[ \tilde{\phi}_{1} = \sqrt{\left( \phi'_{1, a} \right)^{2} + \left( \phi'_{1, z} \right)^{2}} \]
\[ \tilde{\omega}_{2} = \sqrt{\left( \omega_{2, a} \right)^{2} + \left( \omega_{2, z} \right)^{2}} \]
\[ \tilde{\phi}_{3} = \tan^{-1} \left( \frac{\phi_{3, z}}{\phi_{3, a}} \right) \]

and \( \delta_{1} = \tan^{-1} \left( \frac{\phi'_{1, z}}{\phi'_{1, a}} \right) \), \( \delta_{2} = \tan^{-1} \left( \frac{\omega_{2, z}}{\omega_{2, a}} \right) \)
For all of the unstable wave perturbations, the amplitudes are growing exponentially with time. But their phases with respect to the arbitrarily fixed perturbation \( \frac{1}{3} \) remain same with time. Now we can represent these perturbations as functions of \( x \) at any moment of time \( t \) in a figure. This figure will reveal us the structure of the unstable disturbance. For better knowledge of the structure of the disturbance, let us derive the other perturbation parameters such as \( \psi' \) and \( T' \) with the aid of the known perturbation functions (4.6).

Since \( \psi' \) and \( \psi_3' \) are geostrophic their phases are automatically fixed with respect to geopotential perturbation. Because

\[
\frac{\partial \psi_1'}{\partial x} = -A_e \frac{\partial \phi_1'}{\partial x} = \frac{\partial}{\partial x} \left( -A_e \phi_1' \right) = A_e \phi_1' \sin (\mu x + \delta_1)
\]

and

\[
\frac{\partial \psi_3'}{\partial x} = -A_e \frac{\partial \phi_3'}{\partial x} = \frac{\partial}{\partial x} \left( -A_e \phi_3' \right) = A_e \phi_3' \sin \mu x
\]

Thus the geopotential perturbation field is 90° ahead of the meridional velocity field, and the curves of maximum and minimum values of \( \phi_1' \) coincide with the curves of \( \psi_3 = 0 \). Therefore there is no necessity to show these perturbations (\( \psi_1' \) and \( \psi_3' \)) along with the geopotential perturbations.

In our model temperature perturbation is defined at 500 mb through hydrostatic approximation.

\[
T_2' = \frac{1}{R} (\phi_1' - \phi_3')
\]

From (4.6) \( T_2' \) can be written as

\[
A_e \frac{T_2}{\phi_1} \cos (\mu x + \delta_3)
\]

where

\[
\frac{T_2}{\phi_1} = \frac{1}{R} \left\{ (\phi_1')^2 + (\phi_3')^2 - 2 \phi_1' \phi_3' \cos \delta_3 \right\}^{1/2}
\]

and

\[
\delta_3 = \tan^{-1} \left\{ \frac{\phi_1' \sin \delta_3}{\phi_1' \cos \delta_3 - \phi_3'} \right\}
\]
Figures (4.6) and (4.7) represent all the perturbations as harmonic waves with corresponding scales for amplitude on y axis and zonal distance in radians on x-axis (2\pi radians correspond to one wavelength). Figures show the vertical distribution of geopotential at 250 mb and 750 mb, temperature and vertical \( v \) velocity perturbations at 500 mb for two cases. Case (i) (Fig. 4.6) is the structure of the most unstable disturbance of wavelength 6000 km at latitude 45° N with basic wind shear of 6 m sec\(^{-1}\) (100 mb\(^{-1}\)) (3 m sec\(^{-1}\) km\(^{-1}\) or \( U_x = 15 \) m sec\(^{-1}\)) and static stability of 4 MTS units. Case (ii) Fig. 4.7 is the structure of the most unstable disturbance of wavelength 5500 km at latitude 15° N with basic wind shear of 8 m sec\(^{-1}\) (100 mb\(^{-1}\)) (\( U_x = 20 \) m sec\(^{-1}\)) and static stability of 0.5 MTS units. Primarily this figure is intended for later comparison with the equivalent scale of disturbance at the same latitude with primitive equation model.

In figures (4.6) and (4.7) the geopotential disturbance at 500 mb (\( \phi_2' \)) can also be visualized easily with the help of \( \phi_1' \) and \( \phi_3' \) (since \( \phi_2' = \frac{1}{2} (\phi_1' + \phi_3') \)). The air is warmer ahead of the geopotential trough line and colder ahead of the geopotential ridge line. This causes a westward tilt of the geopotential wave with height in accordance with the hydrostatic relation. This tilt of the equal phase line in the vertical makes the sensible heat to be transported northward by the disturbance (since the lag of thermal wave with geopotential wave gives a positive correlation between \( \phi_2' \) and \( \tau_2' \)). Ascending motion takes place in eastern part of the geopotential trough line. Descending motion occurs in eastern part.
\( S_2 = 4 \text{ mts. units at } 45^\circ \text{N}, \quad U_* = 15 \text{ m/s} \)

\[ L = 6000 \text{ km} \]

**FIG. 4.6** STRUCTURE OF THE MOST UNSTABLE DISTURBANCE
\[ \lambda = 0.5 \text{ MTS. UNITS AT } 15^\circ \text{N, } U_* = 20 \text{ m/s} \]
\[ L = 5500 \text{ KM} \]

FIG. 4.7 STRUCTURE OF THE MOST UNSTABLE DISTURBANCE
of the geopotential ridge line. This sort of distribution of $\omega'$ relative to geopotential disturbance is generally expected for unstable disturbance from $\omega'$ equation (4.2). Combined with the distribution of vertical velocity the temperature perturbation structure is such that direct thermal circulation with warm air rising and cold air sinking is taking place. We know the above processes are responsible for the growth of the disturbance with time. So the structure of unstable disturbance chiefly characterized by (a) westward tilt of the geopotential perturbation with height (b) lag of thermal wave with geopotential wave and (c) negative correlation between $\omega'$ and $\tau'$. This sort of structure for unstable disturbance is in agreement with the studies of Charney (1947) and Kuo (1952). The general structure of these unstable disturbances is rather similar to that observed developing disturbances in the atmosphere chiefly in middle latitude region.

Table 4.3 gives the amplitudes and phases of the perturbations computed for cases (i) and (ii).

From Table 4.3 it is worthwhile to note that the amplitude $\tilde{\phi} A_t$ is larger than $\hat{\phi} A_t$. This gives $\frac{\partial \omega'}{\partial \beta}$ negative.
## Table 4.3

**Amplitudes and phases of the perturbation**

**Parameters**

Case (1) Unstable disturbance of wavelength 6000 km, at Latitude 45° with static stability of 4 MTS units and with basic wind shear of 6 m sec\(^{-1}\) (100 mb)\(^{-1}\)

Case (2) Unstable disturbance of wavelength 5500 km, at latitude 15° with static stability of 0.5 MTS units and with basic wind shear of 8 m sec\(^{-1}\) (100 mb)\(^{-1}\)

<table>
<thead>
<tr>
<th>Case No.</th>
<th>Amplitudes and phases of the perturbations</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(\phi_3) m(^2) sec(^{-1})</td>
</tr>
<tr>
<td>1</td>
<td>((A_t)) 50</td>
</tr>
<tr>
<td>Phase</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>((A_t)) 50</td>
</tr>
<tr>
<td>Phase</td>
<td>0</td>
</tr>
</tbody>
</table>

If \(t = 10^5\) secs (\(\approx\) 1 day) \(A_t = 1.99\) for case 1 and \(A_t = 2.77\) for case 2
4.4 Energy transformations for disturbances:

Let us visualise the above processes (section 4.3) for the same cases quantitatively by calculating every possible energy transformation functions for this two level quasigeostrophic model.

We have not shown earlier the energy transformation functions for quasigeostrophic model, but it can be done on the lines of Chapter II.

\[
\frac{\partial E}{\partial t} = I \left[ \frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial z} \right] P, P' \quad (4.7)
\]

\[
\frac{\partial P'}{\partial t} = I \left[ \frac{\partial}{\partial z} P' - \frac{\partial}{\partial z} P \right] - I \left[ \frac{\partial}{\partial z} P, P' \right] \quad (4.8)
\]

The absence of transformation between $P'$ and $P$ is due to lack of meridional dependency of the perturbations and zonal basic current.

Let us express the energy transformations per unit zonal length and per unit width polewards, but extending from the lower to the upper boundary of the atmosphere.

Then

\[
I \left[ \frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial z} \right] P, P' = \int_{\text{base}}^{\text{top}} \omega \frac{\partial^2}{\partial z} \frac{\partial^2 \phi'}{\partial z^2} \, dm \quad (4.9)
\]

\[
I \left[ \frac{\partial}{\partial z} P' - \frac{\partial}{\partial z} P \right] = \int_{\text{base}}^{\text{top}} \frac{1}{\omega} \frac{\partial}{\partial z} \frac{\partial^2 \phi'}{\partial z^2} \, dm \quad (4.10)
\]

Where

\[ dm = \frac{d \phi}{q} \times (1 \times 1) \]
Energy transformation functions (4.9) and (4.10) are defined similarly to that of Chapter II viz. energy flows from left to right quantity in a bracket, if the transformation function takes a positive sign. (4.9) and (4.10) can be written for two level model with the same assumptions such as $\theta_2^{'1} = \frac{1}{\lambda} (\theta_1^{'1} + \theta_3^{'1})$ and $\omega_1^{'1} = \omega_3^{'1} = \frac{1}{\lambda} \omega_2^{'1}$, as we made in Chapter III. And then utilising harmonic wave functions (4.6) for the perturbations we can write (4.9) and (4.10) as

$$I [ P, K'] = \frac{1}{j} A_k^2 \left[ \tilde{\omega}_2 \tilde{\omega}_3 \cos \tilde{\delta}_2 \tilde{\omega}_2 \tilde{\omega}_3 \cos (\tilde{\delta}_2 - \tilde{\delta}_3) \right] (4.11)$$

$$I [ \bar{P}, P'] = \frac{1}{j} A_k^2 \left[ \frac{1}{\lambda^2} \mu \frac{2u_{x_0} \tilde{f}_2 \tilde{f}_3 \sin \tilde{\delta}_1}{\tilde{\delta}_1} \right] (4.12)$$

Let us examine these energy transformation functions over the complete spectrum of unstable waves pertaining to two cases:

Case (i) At latitude $45^0N$ with static stability of 4 MTS units and with basic wind shear of $6 \text{ m sec}^{-1} (100 \text{ mb})^{-1}$ (or $3 \text{ m sec}^{-1} \text{ km}^{-1}$), the spectrum of unstable waves extend from 5000 km to 8000 km from figure (4.3). Case (i) of the previous section is that of the most unstable disturbance of this particular wave spectrum only.

Case (ii) at latitude $15^0N$ with static stability of 0.5 MTS units and with basic wind shear of $6 \text{ m sec}^{-1} (100 \text{ mb})^{-1}$ (or $4 \text{ m sec}^{-1} \text{ km}^{-1}$), the spectrum of unstable waves extend from 4500 km to 8000 km from figure (4.5). The case (ii) of the previous section is that of the most unstable disturbance of this particular wave spectrum only.

Now the system of equations (4.3) is solved for amplitude
functions and phases of the perturbations for the spectrum of waves of case (i) and case (ii) utilizing their corresponding complex eigenvalues $\Lambda$ and associated basic state parameters, on similar lines of section (4.5). Here the amplitudes of the perturbation parameters for each wavelength are computed by evaluating the corresponding amplification parameter $A_t$ at time $t = 10^5$ secs (approximately one day). Making use of these amplitudes and phases of the perturbations, the energy transformation functions (4.11) and (4.12) are calculated for each wavelength of the cases (i) and (ii) and represented as functions of wavelength in figures (4.8) and (4.9) respectively. As expected from equations (4.7) and (4.8) for the unstable disturbances the quantities $I[\bar{p}, p']$ and $I[p', k']$ are positive. Accordingly the kinetic energy of the disturbance grows with time due to the effect of release of eddy available potential energy through the process of direct thermal circulation (see equation 4.7, and expression 4.9). The loss of eddy available potential energy due to this process is supplied by transfer of zonal mean available potential energy (see equation 4.8). In figures 4.8 and 4.9, $I[\bar{p}, p']$ exceeds $I[p', k']$ for all wavelengths.

This shows an increase of eddy available potential energy with time due to partial conversion of its supply into eddy kinetic energy (see equation 4.8). From equation 4.7 the maximum conversion of eddy available potential energy to eddy kinetic energy (i.e. maximum value of $I[p', k']$) corresponds to the maximum rate of increase of kinetic energy of the disturbance with time. Thus the
Fig. 4.8 Energy transformation functions for spectrum of unstable waves with $\frac{s_2}{\alpha} = 4$ mts units & wind shear of 6 msec$^{-1}$ (100 mb)$^{-1}$ at 45$^\circ$ latitude.
FIG. 4-9 ENERGY TRANSFORMATION FUNCTIONS FOR SPECTRUM OF UNSTABLE WAVES WITH $S_2 = 0.5$ MTS UNITS & WIND SHEAR OF 8 mSec$^{-1}$ (100 mb)$^{-1}$ AT 15° LATITUDE
disturbance of wavelength for which the value of \( I \left[ P, k' \right] \) is maximum, is known to be the most unstable disturbance. The wavelengths of the most unstable disturbances of case (i) and case (ii) coincide with those from the stability diagrams (4.3) and (4.5) for corresponding basic wind shears respectively. This coincidence provides the physical and energetical background for the instability of disturbances.

The computed energy transformation functions (4.11) and (4.12) for the most unstable disturbance of cases (i) and (ii) are given in Table (4.4)

**TABLE 4.4**

Energy transformation functions for unstable disturbances

<table>
<thead>
<tr>
<th>Case (i)</th>
<th>The most unstable disturbance of wavelength 6000 km, at latitude 45° with static stability of 4 MTS units and with basic wind shear of 6 m sec(^{-1}) (100 mb)(^{-1}).</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case (ii)</td>
<td>The most unstable disturbance of wavelength 5500 km, at latitude 15° with static stability of 0.5 MTS units and with basic wind shear of 8 m sec(^{-1}) (100 mb)(^{-1}).</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Wavelength</th>
<th>( I \left[ \bar{P}, P' \right] _{k_j} ) sec(^{-1})</th>
<th>( I \left[ P', k' \right] , k'_j ) sec(^{-1})</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case (i)</td>
<td>6000 km</td>
<td>( 1.08 \times 10^{-4} )</td>
</tr>
<tr>
<td>Case (ii)</td>
<td>5500 km</td>
<td>( 21.9 \times 10^{-4} )</td>
</tr>
</tbody>
</table>
\( I[\vec{v},\vec{p}] \) being positive for the unstable disturbances, we may note from expression 4.12 with basic wind shear \( \frac{\partial u}{\partial y} \) negative (\( u > u_0 \) i.e. \( u_* \) is positive), the sign of \( \sin \delta_1 \) must be positive. So the phase angle \( \delta_1 \) is positive and lies between \( 0^\circ \) to \( 180^\circ \).

Thus with positive \( \delta_1 \), (see expression 4.6), the geopotential perturbation at 250 mb will be lagging the corresponding one at 750 mb. Thereby for unstable disturbances a westward tilt of the equal phase line in the vertical for geopotential perturbation is seen from the view of point of energetics (Figures 4.6 and 4.7). Moreover, we can say that the northward transport of sensible heat by the disturbance is uniquely related with the westward tilt of the geopotential perturbation in the vertical (from expression 4.10 and 4.12).

For neutral disturbance (i.e. by definition \( \Delta z = 0 \)) the amplitude functions \( \hat{p}, \hat{p}_3 \) and \( \hat{\omega}_z \) are real from expressions (4.4). And there will be no tilt in the vertical for the geopotential perturbation because \( \tan \delta_1 = 0 \) from expressions (4.6) and \( \omega_z \) wave lags \( \phi'_3 \) wave by \( 90^\circ \) (see \( \tan \delta_2 \) from expression 4.6). Thus with \( \delta_1 = 0 \) and \( \delta_2 = 90^\circ \) both the energy transformation functions
\( I[\vec{p},\vec{p}'] \) and \( I[\vec{p}',\vec{k}'] \) are zero. This explains the nature of neither growth nor decay (\( \frac{\partial k'}{\partial t} = 0; \frac{\partial p'}{\partial t} = 0 \)) for the neutral disturbances.

Finally with a knowledge of vertical structure and energetics for unstable and neutral disturbances (sections 4.3 and 4.4) it is possible to discuss qualitatively the situation of damping disturbances also.
The various physical processes occurring in the disturbances in a statically stable atmosphere with basic zonal wind increasing with height is given in Table 4.5
### Table 4.5
Structure of the Disturbances

<table>
<thead>
<tr>
<th>Type of disturbance</th>
<th>Nature of process</th>
<th>Relative distributions of perturbations</th>
</tr>
</thead>
<tbody>
<tr>
<td>Amplified</td>
<td>Warm air ascent ahead of trough line of ( \phi_2 ) and cold air descent ahead of ridge line of ( \phi_1 ). ( \frac{\omega_2 T_2'}{T_2'} &gt; 0 ); ( \frac{\omega_2 T_2'}{T_2'} &lt; 0 )</td>
<td>( \phi_2 ) wave lags ( \phi_2' ) wave by 90°. ( \phi_1 ) wave lags ( \phi_3 ) wave (westward tilt in the vertical) ( \delta \in (0^\circ, 180^\circ) ). ( \omega_2 ) wave is out of phase with ( \phi_2 ) wave. ( \phi_2 ) wave lags ( \phi_2' ) wave. ( T_2 ) wave lags ( \phi_3 ) wave. 180° &lt; ( \delta_1 ) &lt; 360°. 30° &lt; ( \delta_3 ) &lt; 180°. ( \delta_1 = 90^\circ ).</td>
</tr>
<tr>
<td>Damped (a)</td>
<td>Cold air ascent ahead of trough line of ( \phi_2' ) and warm air descent ahead of ridge line of ( \phi_1' ). ( \frac{\omega_2 T_2'}{T_2'} &lt; 0 ); ( \frac{\omega_2 T_2'}{T_2'} &gt; 0 )</td>
<td>( \phi_2 ) wave lags ( \phi_2' ) wave by 90°. ( \phi_1' ) wave leads ( \phi_3 ) wave (eastward tilt in the vertical) ( \delta \in (180^\circ, 360^\circ) ). ( \omega_2 ) wave is out of phase with ( \phi_2' ) wave. ( \phi_2' ) wave leads ( \phi_2 ) wave. ( T_2 ) wave is out of phase with ( \phi_2' ) wave. 180° &lt; ( \delta_2 ) &lt; 270°. 30° &lt; ( \delta_3 ) &lt; 180°. ( \delta_2 = 270^\circ ).</td>
</tr>
<tr>
<td>Neutral</td>
<td>( \frac{\omega_2 T_2'}{T_2'} = 0 ); ( \frac{\omega_2 T_2'}{T_2'} = 0 )</td>
<td>( \phi_2 ) wave lags ( \phi_2' ) wave by 90°. ( \phi_1' ) wave is in phase with ( \phi_3 ) wave (No tilt in the vertical) ( \delta = 0 ). ( \omega_2 ) wave is out of phase with ( \phi_2 ) wave. ( \delta_2 = 270^\circ ). ( \phi_2' ) wave is out of phase with ( \phi_2 ) wave (warm low &amp; cold high) ( \delta_3 = 180^\circ ). ( T_2 ) wave is out of phase with ( \phi_2 ) wave (cold low &amp; warm high) ( \delta_3 = 0 ).</td>
</tr>
</tbody>
</table>