Chapter 2

Thermal Instability of Rivlin-Ericksen Elastico-Viscous Fluid Permeated with Suspended Particles in Hydrodynamics in a Porous Medium


2.1 Introduction

The formulation and derivation of the basic equations of a layer of a fluid heated from below in a porous medium, using the Boussinesq approximation, has been given in the treatise by Joseph [24]. When a fluid permeates an isotropic and homogeneous porous medium, the gross effect is represented by Darcy’s law. The study of a layer of a fluid heated from below in a porous medium is motivated both theoretically and by its practical applications in engineering. Among the applications in engineering disciplines one can find the food process industry, chemical process industry, solidification and centrifugal casting of metals. The development of geothermal power resources has increased general interest in the properties of convection in porous media, Singh and Gupta [25].

A comprehensive account of the effect of a uniform magnetic field on the layer of a Newtonian fluid heated from below was given by Chandrasekhar [26]. The effect of a magnetic field on the stability of the fluid flow is of interest in geophysics, particularly in the study of earth core where the earth’s mantle, which consists of a conducting fluid, behaves like a porous medium which can become convectively unstable as a result of differential diffusion. The results of flow through a porous medium in the presence of a magnetic field are applied in the study of the stability of a convective flow in the geothermal region. Lapwood [27] studied the stability of a convective flow in hydrodynamics using Rayleigh’s procedure. Wooding [28] considered the Rayleigh instability of a thermal boundary layer in the flow through a porous medium.

The fluid may not be absolutely pure but may, instead, be permeated with suspended (or dust) particles. The effect of particle mass and heat capacity on the onset of Bénard convection was considered by Scanlon and Segel [23]. The effect of suspended particles was found to destabilize the layer. In another context, Palaniswamy and Purushotham [29] studied the stability of a shear flow of stratified fluids with fine dust and found the effect of fine dust to increase the region of instability. The thermal instability of fluids in a porous medium in the presence of suspended particles was studied by Sharma and Sharma [30]. The suspended particles and the permeability of the medium were found to destabilize the layer. Sharma and Kumar [31] studied the Rayleigh-Taylor instability of fluids in porous media in the
presence of suspended particles and variable magnetic field. In all the above studies, the fluid has been considered to be Newtonian. One such class of elastico-viscous fluids is the Rivlin-Ericksen fluid [32]. Srivastava and Singh [33] studied the unsteady flow of the dusty elastico-viscous Rivlin-Ericksen fluid through channels of different cross sections in the presence of a time-dependent pressure gradient. In other study, Garg et al. [34] studied the rectilinear oscillations of a sphere along its diameter in a conducting dusty Rivlin-Ericksen fluid in the presence of a uniform magnetic field. Sharma and Kumar [35] studied the thermal instability of a layer of a Rivlin-Ericksen elastico-viscous fluid in the presence of suspended particles. In another study, Kumar [36] considered the stability of suspended Rivlin-Ericksen elastico-viscous fluids permeated with suspended particles in a porous medium. It is this class of elastico-viscous fluids we are particularly interested in studying the effect of suspended or dust particles on the Rivlin-Ericksen elastico-viscous fluid heated from below in a porous medium in the presence of a uniform horizontal magnetic field.

2.2 Formulation of the Problem

Let us consider the following physical quantities for the formulation of the problem. Tensor quantities like stress, rate of strain, shear stress, Kronecker delta be represented by $T_{kl}$, $e_{kl}$, $\tau_{kl}$ and $\delta_{kl}$ respectively. Vector quantities like velocity and position vector be represented by $\vec{v}$ and $\vec{x}$ respectively. $p$ denotes the isotropic pressure and material properties viscosity and viscoelasticity be denoted by $\mu$ and $\mu'$. Constitutive relations between the stress and rate of strain for the Rivlin-Ericksen fluid are

$$T_{kl} = -p\delta_{kl} + \tau_{kl},$$

$$\tau_{kl} = \rho \left( \nu + \nu' \frac{\partial}{\partial t} \right) e_{kl},$$

$$e_{kl} = \frac{1}{2} \left( \frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k} \right).$$

(2.1)

In porous medium, an infinite horizontal layer of depth $d$ of an electrically conducting viscoelastic Rivlin-Ericksen fluid which is acted on by gravity force
\( \vec{g}(0, 0, -g) \) and a uniform horizontal magnetic field \( \vec{H}(0, 0, H) \) is considered. For the study of thermal instability, layer is heated from underside and steady adverse temperature gradient \( \beta \) is maintained, where \( \beta = \left| \frac{dT}{dz} \right| \).

Let the fluid properties like pressure, temperature, density, velocity of pure fluid, kinematic viscosity and kinematic viscoelasticity be denoted by \( p, T, \rho, \vec{v}(u, v, w), \nu \) and \( \nu' \) respectively. Properties of suspended particle like velocity and number density be represented by \( u(x, t) \) and \( N(x, t) \). \( \vec{g} \) is the gravitational acceleration, \( \epsilon \) represents the medium porosity and \( k_1 \) represents the medium permeability. \( K = 6\pi\mu\eta' \) is the Stokes’ drag coefficient for the particle having the radius \( \eta' \).

Then the flow governing equations of conservation of mass and momentum in a porous medium for the considered fluid in the presence of magnetic field and suspended particles are

\[
\frac{1}{\epsilon} \left[ \frac{\partial \vec{v}}{\partial t} + \frac{1}{\epsilon}(\vec{v} \cdot \nabla)\vec{v} \right] = -\frac{1}{\rho_0} \nabla p - g \left( 1 + \frac{\delta \rho}{\rho_0} \right) \vec{X} - \frac{1}{k_1} \left( \nu + \nu' \frac{\partial}{\partial t} \right) \vec{v} + \frac{KN}{\rho_0 \epsilon} (\vec{u} - \vec{v}) + \frac{\mu_e}{4\pi \rho_0} \left[ (\nabla \times \vec{H}) \times \vec{H} \right]
\]  

(2.2)

and \( \nabla \cdot \vec{v} = 0 \).  

(2.3)

In the above equations of conservation of momentum (2.2), some assumptions regarding the shape and velocity of the suspended particles are taken as

- Shape of the suspended particles in the fluid is uniform spherical.
- Relative velocities between the fluid and particles is small.
- Large distance between the particles as compare to their diameter. So Interparticle reactions are ignored.
- Gravity, pressure, Darcian force and magnetic field effect on the suspended particles are negligibly small, so ignored.
- Extra force due to the presence of particles is proportional to velocity difference between the particles and the fluid.
- Force exserted by fluid on particles and force exerted by particles on fluid balance each other.
So there must be an extra force equal in magnitude but opposite in sign in the equations of conservation of momentum or motion for the particles. If \( mN \) is the mass of particles per unit volume, then under the above assumptions, equations of conservation of momentum and mass for the particles are

\[
mN \left[ \frac{\partial \vec{u}}{\partial t} + \frac{1}{\epsilon} (\vec{u} \cdot \nabla) \vec{u} \right] = KN(\vec{v} - \vec{u}) \quad \text{and} \quad (2.4)
\]

\[
\epsilon \frac{\partial N}{\partial t} + \nabla \cdot (N \vec{u}) = 0. \quad (2.5)
\]

Let at constant volume, \( C_v \) is the heat capacity of the fluid, \( C_{pt} \) denote the heat capacity of the particles, \( T \) is the temperature and \( q \) is effective thermal conductivity of the pure fluid. If the fluid and the particles are in thermal equilibrium, then equation of heat conduction is

\[
[r_0 C_v \epsilon + \rho_s C_s (1 - \epsilon)] \frac{\partial T}{\partial t} + \rho_0 C_v (\vec{v} \cdot \nabla) T + m N C_{pt} \left( \epsilon \frac{\partial}{\partial t} + \vec{u} \cdot \nabla \right) T = q \nabla^2 T
\]

\[ (2.6) \]

where \( \rho_s \) is the density and \( C_s \) is the heat capacity of the solid matrix. Maxwell’s equations yield:

\[
\epsilon \frac{\partial \vec{H}}{\partial t} = (\vec{H} \cdot \nabla) \vec{v} + \epsilon \eta \nabla^2 \vec{H} \quad \text{and} \quad (2.7)
\]

\[
\nabla \cdot \vec{H} = 0 \quad (2.8)
\]

where \( \eta \rightarrow \) The electrical resistivity. Equation of state for fluid is

\[
\rho = \rho_0 (1 - \alpha \delta T) = \rho_0 [1 - \alpha (T - T_0)] \quad (2.9)
\]

where \( \alpha \rightarrow \) co-efficient of thermal expansion, \( \rho_0 \rightarrow \) density of the fluid at the bottom surface \( z = 0 \) and \( T_0 \rightarrow \) at temperature of the fluid at \( z = 0 \). Initially the system is taken as quiescent layer (no settling) with a uniform particle distribution \( N_0 \). Initial values of the variables are

\( \vec{u} = (0, 0, 0), \quad \vec{v} = (0, 0, 0), \quad N_0 = \text{Constant}, \quad T = -\beta z. \)

which is an exact solution to the governing equations.
2.2.1 Perturbation of Equations

Let \( \delta p \) denote the perturbation in pressure \( p \), \( \delta \rho \) denote the perturbation in density \( \rho \), \( \theta \) denote the perturbation in temperature \( T \), \( \vec{v}(u,v,w) \) denote the perturbation in fluid velocity (zero initially), \( \vec{u}(l,r,s) \) denote the perturbation in particle velocity (zero initially), \( N \) denote perturbations in suspended particles number density \( N_0 \) and \( \vec{h}(h_x, h_y, h_z) \) denote perturbations in magnetic field \( \vec{H}(0,0,H) \). Since density is depends upon the temperature, so perturbation in temperature will bring change is density defined by the relation \( \delta \rho = -\alpha \rho_0 \theta \).

Governing equations of flow hold true for both the initial and perturbed state. Therefore, linearized perturbed equations of the problem are

\[
\frac{1}{\epsilon} \frac{\partial \vec{v}}{\partial t} = -\frac{1}{\rho_0} \nabla \delta p - g \alpha \theta \lambda - \frac{1}{k_1} \left( \nu + \nu' \frac{\partial}{\partial t} \right) \vec{v} + \frac{K N}{\rho_0} (\vec{u} - \vec{v}) + \frac{\mu_e}{4\pi \rho_0} \left[ (\nabla \times \vec{h}) \times \vec{H} \right],
\]

\( (2.10) \)

\[
\nabla . \vec{v} = 0,
\]

\( (2.11) \)

\[
mN_0 \frac{\partial \vec{u}}{\partial t} = KN_0 (\vec{v} - \vec{u}),
\]

\( (2.12) \)

\[
(E + \epsilon^2) \frac{\partial \theta}{\partial t} = \beta (w + h_s) + \kappa \nabla^2 \theta,
\]

\( (2.13) \)

\[
\epsilon \frac{\partial \vec{h}}{\partial t} = (\vec{H} \nabla) \vec{v} + \epsilon \eta \nabla^2 \vec{h}
\]

\( (2.14) \)

and \( \nabla . \vec{h} = 0 \)

\( (2.15) \)

where \( E = \epsilon + (1 - \epsilon) \frac{\rho_s C_s}{\rho_0 C_v} \), \( h = \frac{mN_0 C_p}{\rho_0 C_v} \) and \( \kappa = \frac{q}{\rho_0 C_v} \).

Eliminating \( u \) in equation (2.10) by using equation (2.12), write the resulting equation in scalar components eliminate \( u, v, \delta p, h_x, h_y \) between them, with the help of equations (2.11) and (2.15), we obtain

\[
n' \nabla^2 w + \frac{\epsilon}{k_1} \left( \nu + \nu' \frac{\partial}{\partial t} \right) \nabla^2 w - \epsilon g \alpha \left( \frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} \right) - \frac{\mu_e H}{4\pi \rho_0} \frac{\partial}{\partial x} \nabla^2 h_z = 0,
\]

\( (2.16) \)
\( \left( \frac{m}{K} \frac{\partial}{\partial t} + 1 \right) \left[ (E + h\epsilon) \frac{\partial}{\partial t} - \kappa \nabla^2 \right] \theta = \beta \left[ \left( \frac{m}{K} \frac{\partial}{\partial t} + 1 \right) + h \right] w \) and \( (2.17) \)

\[ \epsilon \left[ \frac{\partial}{\partial t} - \eta \nabla^2 \right] h_z = H \frac{\partial w}{\partial x} \] (2.18)

where \( n' = \frac{\partial}{\partial t} \left[ 1 + \frac{m_N_0 K}{\rho_0} \frac{\partial}{\partial t} + K \right] \)

### 2.3 Dispersion relation

Perturbed quantities are assumed to be of the following form for the analysis of disturbances into normal modes

\[ [w, \theta, h_z] = [W(z), \Theta(z), X(z)] \exp(ik_x x + i k_y y + nt) \] (2.19)

where \( k_x \) is the wave number along \( x \)-direction, and \( k_y \) is wave number along \( y \)-direction. \( k = \sqrt{k_x^2 + k_y^2} \) = resultant wave number and \( n = \) growth rate = complex constant in general. Using expression (2.19), equations (2.16)-(2.18) in a non-dimensional form become

\[ \left[ \frac{\sigma'}{\epsilon} + \frac{1}{p_1} (1 + F \sigma) \right] (D^2 - a^2) W + \frac{g \alpha a^2 \Theta}{\nu} - \frac{ik_x \mu_e H_d^2}{4 \pi p_0 \nu} (D^2 - a^2) X = 0, \] (2.20)

\[ \left[ \tau \nu \sigma \frac{d^2}{d^2} + 1 \right] \left[ (D^2 - a^2) - (E + h \epsilon)p_3 \sigma \right] \Theta = -\frac{\beta d^2}{\kappa} \left[ H' + \frac{\tau \nu \sigma}{d^2} \right] W \] (2.21)

and \[ \left[ (D^2 - a^2) - p_2 \sigma \right] \chi = -\frac{ik_x H_d^2}{\epsilon \eta} W \] (2.22)

where the co-ordinates \( x, y, z \) have expressed in the new unit of length \( d \), time \( t \) in the new unit of length \( d^2 \) and put \( a = kd, \sigma = \frac{\nu d^2}{\nu}, p_3 = \frac{\nu}{\kappa} \rightarrow \) Prandtl number, \( p_2 = \frac{\nu}{\eta} \rightarrow \) magnetic Prandtl number, \( p_1 = \frac{k_1 d^2}{k} \rightarrow \) dimensionless medium permeability, \( F = \frac{\nu}{d^2} \rightarrow \) dimensionless kinematic viscoelasticity, \( \sigma' = \frac{\nu d^2}{\nu} \), \( H' = h + 1, \tau = \frac{mn}{K d^2} \) and \( D = \frac{d}{dz} \).
By eliminating $X$ and $\Theta$ between equations (2.20)-(2.22), we obtain

$$
\left[1 + \frac{\tau \nu \sigma}{d^2}\right] \left[(D^2 - a^2) - (E + h\epsilon)p_3\sigma\right] \left\{\left[\frac{\sigma'}{\epsilon} + \frac{1}{p_1}(1 + F\sigma)\right] \left[(D^2 - a^2) - p_2\sigma\right]
- \frac{k_x^2Q}{\epsilon}\right\} (D^2 - a^2) W = Ra^2 \left[H' + \frac{\tau \nu \sigma}{d^2}\right] \left[(D^2 - a^2) - p_2\sigma\right] W \quad (2.23)
$$

where $R = \frac{\gamma \alpha \beta d}{\nu \kappa}$ = Rayleigh number and $Q = \frac{\mu H^2 d^2}{4 \pi \rho \nu \eta}$ = Chandrasekhar number.

The boundary conditions, suitable for the problem, are Chandrasekhar [26]. For the solution to the problem, free boundaries are considered which is little artificial in nature. Also Temperatures at the boundaries are kept fixed and the medium adjoining the fluid is perfectly conducting.

$$W = 0, \quad D^2 W = 0, \quad \Theta = 0, \quad X = 0 \quad at \quad z = 0 \quad and \quad z = 1. \quad (2.24)$$

Obviously that the even order derivatives of $W$ vanish on the boundaries and hence the proper solution of $W$ characterizing the lowest mode is

$$W = W_0 \sin(\pi z) \quad (2.25)$$

where $W_0 = Constant$. By putting the solution (2.25) in equation (2.23), the dispersion relation can be written as

$$R_1 = \left(1 + x\right) \left[H' + \frac{i \nu \tau \pi^2 \sigma}{d^2}\right] \left\{(1 + x) + i\sigma_1 p_2\right\} + \frac{Q_1 \cos^2 \theta}{\epsilon} 
\left[1 + \frac{\nu \tau \pi^2 \sigma}{d^2}\right] \left[H' + \frac{i \nu \tau \pi^2 \sigma}{d^2}\right] \left\{(1 + x) + i\sigma_1 p_2\right\} \quad (2.26)
$$

where $x = \frac{a^2}{\pi^2}$, $\sigma_1 = \frac{\sigma}{\pi^2}$, $P = \pi^2 p_1$ and $R_1 = \frac{R}{\pi^4}$,

$$i\sigma_1' = \frac{\sigma'}{\pi^2}, \quad Q_1 = \frac{Q}{\pi^2}, \quad and \quad k_x = k\cos \theta.$$

### 2.4 Stationary convection

Put $\sigma = 0$, for stationary convection, and the dispersion relation (2.26) becomes

$$R_1 = \left(1 + x\right) \left[\frac{1 + x + Q_1 \cos^2 \theta}{\epsilon} \frac{1}{x H'}\right] \quad (2.27)$$
Thus, it is found that for stationary convection the viscoelastic parameter $F$ vanishes with $\sigma$ and the Rivlin-Ericksen elastico-viscous fluid behaves like an ordinary Newtonian fluid. To study the effects of the magnetic field, suspended particles and medium permeability, we examine the nature of $\frac{dR_1}{dQ_1}$, $\frac{dR_1}{dH'}$, and $\frac{dR_1}{dP}$, Equation (2.27) yields:

$$\frac{dR_1}{dQ_1} = \frac{(1 + x)\cos^2 \theta}{H'\epsilon},$$  \hspace{1cm} (2.28)

$$\frac{dR_1}{dH'} = -\frac{(1 + x) \left[ \frac{1+x}{P} + \frac{Q_1 \cos^2 \theta}{\epsilon} \right]}{xH'^2} \hspace{1cm} (2.29)$$

and

$$\frac{dR_1}{dP} = -\frac{(1 + x)^2}{xH'^2P^2}. \hspace{1cm} (2.30)$$

Which shows that the magnetic field has a stabilizing effect whereas the suspended particles and medium permeability have a destabilizing effect on thermal convection in the Rivlin-Ericksen fluid permeated with suspended particles in a porous medium in hydrodynamics for stationary convection. Graphically, we analyse the magnetic field, suspended particles and medium permeability as follows:

![Figure 2.1: The variation of dimensionless Rayleigh number ($R_1$) with wave number ($x$), for $H' = 10, P = 2, \theta = 45^0, \epsilon = 0.5$ and $Q_1 = 25, 50, 75.$](image_url)
In Figure 2.1 (refer table 1), we have $x = 1, 2, 3, 4, 5, 6$ and $H' = 10$, $P = 2, \epsilon = 0.5$, $\theta = 45^0$ and $Q_1 = 25, 50, 75$, found that, if magnetic field is increased growth rate is also increased, shows the effect of stabilization on the system.

Whereas in Figure 2.2 (refer table 2), $x = 1, 2, 3, 4, 5, 6$, $H' = 5, 10, 15$, $P = 2$ and $\epsilon = 0.5, \theta = 45^0, Q_1 = 25$, shows that, if suspended particles are increased growth rate is decreased, gives the effect of destabilizing effect on the system.

In Figure 2.3 (refer table 3), by using values of $H' = 10, \epsilon = 0.5, \theta = 45^0, Q_1 = 25$, $P = 0.1, 0.2, 0.6$ and $x = 1, 2, 3, 4, 5, 6$, found that, when medium permeability is increased, growth rate is decreased, gives the destabilizing effect on the system.
2.5 Stability of the system of oscillatory modes

Multiplying equation (2.20) by the complex conjugate of $W$, i.e., $W^*$, integrating over the range of $z$ from $z = 0$ to $z = 1$ and making use of equations (2.21) and (2.22) together with the given physical boundary conditions (2.24), we obtain

\[
\left[ \frac{\sigma'}{\epsilon} + \frac{1}{p_1} (1 + F \sigma) \right] I_1 - \frac{g \alpha \kappa a^2}{\nu \beta} \left[ \frac{d^2 + \nu \tau \sigma^*}{H' d^2 + \nu \tau \sigma} \right] [I_2 + (E + \hbar \epsilon) p_3 \sigma^* I_3] + \frac{\mu \epsilon \eta}{4 \pi \rho_0 \nu} [I_4 + p_2 \sigma^* I_5] = 0 \quad (2.31)
\]

where

\[
I_1 = \int_0^1 (|DW|^2 + a^2 |w|^2) \, dz, \quad I_2 = \int_0^1 (|D\Theta|^2 + a^2 |\Theta|^2) \, dz,
\]

\[
I_3 = \int_0^1 |\Theta| \, dz, \quad I_4 = \int_0^1 (|D^2 X|^2 + 2a^2 |DX|^2 + a^4 |X|^2) \, dz,
\]

\[
I_5 = \int_0^1 (|DX|^2 + a^2 |X|^2) \, dz
\]

and $\sigma^* \to$ complex conjugate of $\sigma$. The integrals $I_1, I_2, I_3, I_4, I_5$ are all positive definite. Putting $\sigma = i \sigma, f = \frac{m N_0}{\hbar}$, and by equating the imaginary parts of equation (2.31), we obtain

\[
\sigma_i \left\{ \frac{1}{\epsilon} \left( 1 + \frac{f}{1 + p_3^2 \tau^2 \sigma_i^2} \right) + \frac{F}{p_1} \right\} I_1 + \frac{g \alpha \kappa a^2}{\nu \beta (H'^2 d^4 + \nu^2 \tau^2 \sigma_i^2)} \left\{ d^2 \nu \tau \hbar I_2 + p_3 (E + \hbar \epsilon) (H'^2 d^4 + \nu^2 \tau^2 \sigma_i^2) I_3 \right\} + \frac{\mu \epsilon \eta \hbar p_2}{4 \pi \rho_0 \nu} I_5 = 0. \quad (2.32)
\]

Equation (2.32) yields that $\sigma_i = 0$ or $\sigma_i \neq 0$, which means that modes may be non-oscillatory or oscillatory. In the absence of the magnetic field, equation (2.32) is reduced to

\[
\sigma_i \left\{ \frac{1}{\epsilon} \left( 1 + \frac{f}{1 + p_3^2 \tau^2 \sigma_i^2} \right) + \frac{F}{p_1} \right\} I_1 + \frac{g \alpha \kappa a^2}{\nu \beta (H'^2 d^4 + \nu^2 \tau^2 \sigma_i^2)} \left\{ d^2 \nu \tau \hbar I_2 + p_1 (E + \hbar \epsilon) (H'^2 d^4 + \nu^2 \tau^2 \sigma_i^2) I_3 \right\} = 0. \quad (2.33)
\]

Thus, $\sigma_i = 0 \Rightarrow$ the principle of exchange of stabilities is valid but oscillatory modes are not allowed. Whereas the quantity inside the brackets is positive definite. The presence of the magnetic field introduces oscillatory modes.
2.6 Conclusion

Presence of Magnetic field showed the stabilizing effect whereas presence of suspended particles and medium permeability showed the destabilizing effect in the study of Rivlin-Ericksen fluid.